Pushing to the Top with K-induction

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Agenda

IC3 is one of the most powerful algorithms for model checking safety properties

Very active area of research:

• A. Bradley: *SAT-Based Model Checking Without Unrolling*. VMCAI 2011
  (IC3 stands for “Incremental Construction of Inductive Clauses for Indubitable Correctness”)

• N. Eén, A. Mishchenko, R. Brayton: *Efficient implementation of property directed reachability*. FMCAD 2011
  (PDR stands for “Property Directed Reachability”)

  ...

• In this talk, I present a new IC3-based algorithm, called QUIP
  (QUIP stands for “a QUest for an Inductive Proof”)

• and show how QUIP can be extended with k-induction
A brief preview of Quip

Quip extends IC3 by allowing for

- **A wider range of conjectures (proof obligations)**
  - Designed to push already existing lemmas more aggressively
  - Allows to push a given lemma by learning additional *supporting* lemmas
    (and hopefully to compute an inductive invariant faster)

- **Forward reachable states**
  - Explain why a lemma cannot be pushed
  - Allows to keep the number of proof obligations under control

These are integrated into a single algorithmic procedure

The experimental results look good
Problem: Symbolic Safety and Reachability

A transition system $P = (V, \text{Init}, \text{Tr}, \text{Bad})$  
$P$ is UNSAFE if and only if there exists a number $N$ s.t.

$$\text{Init}(X_0) \land \left( \bigwedge_{i=0}^{N-1} \text{Tr}(X_i, X_{i+1}) \right) \land \text{Bad}(X_N) \not\models \bot$$

$P$ is SAFE if and only if there exists a safe inductive invariant $\text{Inv}$ s.t.

$$\text{Init} \Rightarrow \text{Inv}$$

$$\text{Inv}(X) \land \text{Tr}(X, X') \Rightarrow \text{Inv}(X')$$

$$\text{Inv} \Rightarrow \neg \text{Bad}$$
Inductive Invariants

System S is safe iff there exists an inductive invariant Inv:

- **Initiation:** Initial ⊆ Inv
- **Safety:** Inv ∩ Bad = ∅
- **Consecution:** TR(Inv) ⊆ Inv  
  i.e., if s ∈ Inv and s ↝ t  
  then t ∈ Inv
Inductive Invariants

System S is safe iff there exists an inductive invariant $\text{Inv}$:

- **Initiation:** $\text{Initial} \subseteq \text{Inv}$
- **Safety:** $\text{Inv} \cap \text{Bad} = \emptyset$
- **Consecution:** $\text{TR}(\text{Inv}) \subseteq \text{Inv}$

i.e., if $s \in \text{Inv}$ and $s \leadsto t$ then $t \in \text{Inv}$

System S is **safe** if $\text{Reach} \cap \text{Bad} = \emptyset$
Generalizing from Bounded Proofs

- **A counterexample of length N exists?**
  - **SAT**
  - Yes: T, N=0
  - No: N := N+1

- **Generalize proof**
  - **SAT**
  - **Yes**

- **Is a safe inductive invariant?**
  - **SAT**
  - **YES**

- Candidate Inv
IC3/PDR

\[ F = [\text{Init}] \]

MkSafe

\[ G = [G_0, \ldots, G_N] \]

Push

\[ F = [F_0, \ldots, F_N] \]

Yes \( \exists \ i, F_i = F_{i+1} \) \rightarrow \text{SAFE}

No \rightarrow \text{MkSafe}

CEX
IC3/PDR In Pictures: MkSafe

MkSafe

\[ p \land q \land z \]

\[ !p \]

\[ F_0 \]

\[ F_1 \]

\[ F_2 \]

\[ F_3 \]
IC3/PDR in Pictures: Push

Algorithm Invariants

- $F_i \rightarrow \neg \text{Bad}$
- $\text{Init} \rightarrow F_i$
- $F_i \rightarrow F_{i+1}$
- $F_i \land T \rightarrow F'_{i+1}$

Inductive
A quick review of IC3/PDR

Input:
• A safety verification problem (Init, Tr, Bad)

Output:
• A **counterexample** (if the problem is **UNSAFE**),
• A **safe inductive invariant** (if the problem is **SAFE**)
• Resource Limit

Main Data-structures:
• A current working level \( N \)
• An *inductive trace*
• A set of *proof obligations*
Inductive Trace

Let $F_0, F_1, F_2, \ldots, F_\infty$ be conjunctions of lemmas (in practice, clauses). We say that $F_0, F_1, F_2, \ldots, F_\infty$ is an *inductive trace* if:

1. $F_0 = \text{INIT}$
2. $F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \ldots \Rightarrow F_\infty$ (monotone)
3. $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_\infty$ as sets of lemmas (s. monotone)
4. $F_i \land \text{TR} \Rightarrow F_{i+1}'$ for $i \geq 0$ (including $F_\infty \land \text{Tr} \Rightarrow F_\infty'$). (inductive)

**Remarks:**

This definition is slightly different from the original definition:

- the sequence $F_0, F_1, F_2, \ldots$ is conceptually *infinite* (with $F_i = T$ for all sufficiently large $i$)
- we add $F_\infty$ as the last element of the trace (as suggested in PDR)

Each $F_i$ over-approximates states that are reachable in $i$ steps or less (in particular, $F_\infty$ contains all reachable states)
Safe Monotone Inductive Trace

- $F_i$ over-approximates the states that are reachable in at most $i$ steps
- If $F_{j+1} \Rightarrow F_j$ then $F_j$ is an inductive invariant

\[ \text{Bad } = \neg p \]
Proof Obligations in IC3

A proof obligation in IC3 is a pair \((s, i)\), where
\begin{itemize}
  \item \(s\) is a (generalized) cube over state variables
  \item \(i\) is a natural number (called level)
\end{itemize}

We say that \((s, i)\) is blocked (or that \(s\) is blocked at level \(i\)) if \(F_i \Rightarrow \neg s\).

Given a proof obligation \((s, i)\), IC3 attempts to strengthen the inductive trace in order to block it.

Remarks:
In IC3, \(s\) is identified with a counterexample-to-induction (CTI)

If \((s, i)\) is a proof obligation and \(i \geq 1\), then \((s, i-1)\) is already blocked

All proof obligations are managed via a priority queue:
\begin{itemize}
  \item Proof obligations with smallest level are considered first
  \item (additional criteria for tie-breaking)
\end{itemize}
Recursive Blocking Stage in IC3

// Find a counterexample, or strengthen the inductive trace
// s.t. \( F_N \Rightarrow \neg s \) holds

IC3_recBlockCube(s, N)
Add(Q, (s, N))

while \( \neg \text{Empty}(Q) \) do
  (s, k) \leftarrow \text{Pop}(Q)
  if (k = 0) return "Counterexample"
  if (\( F_k \Rightarrow \neg s \)) continue
  if (\( F_{k-1} \land Tr \land s' \)) is SAT
    t \leftarrow \text{generalized predecessor of } s
    Add(Q, (t, k-1))
    Add(Q, (s, k))
  else
    \( \neg t \leftarrow \text{generalize } \neg s \text{ by inductive generalization (to level } m \geq k) \)
    add \( \neg t \) to \( F_m \)
    if (m<N) Add(Q, (s, m+1))

Pushing stage in IC3

// Push each clause to the highest possible frame up to N
IC3_Push()
    for k = 1 .. N-1 do
        for c ∈ F_k \ F_{k+1} do
            if (F_k ∧ Tr ⇒ c')
                add c to F_{k+1}
        if (F_k = F_{k+1})
            return “Proof” // F_k is a safe inductive invariant
Towards improving IC3 (1)

IC3 is an excellent algorithm! So, what do we want?

We want *more control* on which lemmas to learn:
- Each lemma in the inductive trace is neither an over-approximation nor an under-approximations of reachable states (a lemma in $F_k$ only over-approximates states reachable within $k$ steps):
  - IC3 may learn lemmas that are *too weak* (ex. $C_1$) – prune less states
  - IC3 may learn lemmas that are *too strong* (ex. $C_2$) – cannot be in the inductive invariant
Towards improving IC3 (2)

We want to know if an already existing lemma is good (in $F_\infty$) or bad (e.g., $C_2$ from before):
- Avoid periodically pushing bad lemmas
- Ideally, we also want to prune less useful lemmas

We want to prioritize reusing already discovered lemmas over learning of new ones:
- When the same cube $s$ is blocked at different levels, usually different lemmas are discovered
  - Although, IC3 partially addresses this using pushing (and other optimizations)
- Use the same lemma to block $s$ (at the expense of deriving additional supporting lemmas)
  - Although, in general different lemmas are of different “quality” and having some choice may be beneficial
Immediate improvement: unlimited pushing

// Push each clause to the highest possible frame up to N
IC3_Push_Unlimited()
   for k = 1 .. do
      for c ∈ F_k \ F_{k+1} do
         if (F_k ∧ Tr ⇒ c')
            add c to F_{k+1}
      if (F_k = F_{k+1})
         F_∞ ← F_k
      if (F_∞ ⇒ ¬Bad)
         return “Proof” // F_∞ is a safe inductive invariant

Claim: after pushing F_∞ represents a maximal inductive subset of all lemmas discovered so far

Remark: the idea to compute maximal inductive invariants is suggested in PDR but claimed to be ineffective. In our implementation, “unlimited pushing” leads to ~10% overall speed up.
Pushing is Useful

Why pushing is useful:
• During the execution of IC3, the sets $F_i$ are incrementally strengthened, and so it may happen that $F_k \land \text{TR} \Rightarrow c'$, even though this was not true at the time that $c$ was discovered.

Why pushing is good:
• By pushing $c$ from $F_k$ to $F_{k+1}$, we make $F_k$ more inductive (and if $F_k$ becomes equal to $F_{k+1}$, then $F_k$ becomes an inductive invariant).
• Suppose that $c \in F_k$ blocks a proof obligation $(s, k)$. By pushing $c$ from $F_k$ to $F_{k+1}$, we also block the proof obligation $(s, k+1)$.
• Pushing Clauses = Improving Convergence = Reusing old lemmas for blocking bad states.
What Happens when Pushing Fails

*Why pushing may fail:* suppose that $c \in F_k \setminus F_{k+1}$ but $F_k \land TR$ does not imply $c'$. Why?

There are two alternatives:
1. $c$ is a valid over-approximation of states reachable within $k+1$ steps, but $F_k$ is not strong enough to imply this
   • We can strengthen the inductive trace so that $F_k \land TR \Rightarrow c'$ becomes true

2. $c$ is **NOT** a valid over-approximation of states reachable within $k+1$ steps
   • There is a real *forward reachable* state $r$ that is excluded by $c$
   • $c$ has no chance to be in the safe inductive invariant
   • $c$ is a *bad* lemma

A similar reasoning is used in:
Z. Hassan, A. Bradley, F. Somenzi: *Better Generalization in IC3*. FMCAD 2013
Two interdependent ideas

1. Prioritize pushing existing lemmas
   • Given a lemma \( c \in F_k \setminus F_{k+1} \), we can add \((\neg c, k+1)\) as a may-proof-obligation
     • May-proof-obligations are “nice to block”, but do not need to be blocked
   • If \((\neg c, k+1)\) can be blocked, then \(c\) is pushed to \(F_{k+1}\)
   • If \((\neg c, k+1)\) cannot be blocked, then we discover a concrete reachable state \(r\) that is excluded by \(c\) and that explains why \(c\) cannot be inductive

2. Discover and use new forward reachable states
   • These are an under-approximation of forward reachable states
   • Given a reachable state, all the existing lemmas that exclude it are bad
     • Bad lemmas are never pushed
   • Reachable states may show that certain may-proof-obligations cannot be blocked
   • Reachable states may be used when generalizing lemmas
   • Conceptually, computing new reachable states can be thought of as new Init states
Quip

Input:
• A safety verification problem \((\text{Init, Tr, Bad})\)

Output:
• A counterexample (if the problem is UNSAFE),
• A safe inductive invariant (if the problem is SAFE)
• Resource Limit

Main Data-structures:
• A current working level \(N\)
• An \textit{inductive trace} (same as IC3)
• A set of \textit{proof obligations} (\textit{similar} to IC3)
• A set \(R\) of \textit{forward reachable states}
Proof Obligations in Quip

A proof obligation in Quip is a triple \((s, i, p)\), where
- \(s\) is a (generalized) cube over state variables
- \(i\) is a natural number
- \(p \in \{\text{may, must}\}\)

**Remarks:**
- As in IC3, if \((s, i, p)\) is a proof obligation and \(i \geq 1\), then \((s, i-1)\) is assumed to be already blocked
- As in IC3, all proof obligations are managed via a priority queue:
  - Proof obligations with *smallest level* are considered first
  - In case of a tie, proof obligations with *smallest number of literals* are considered first
  - (additional criteria for tie-breaking)
- Have a “parent map” from a proof obligation to its parent proof obligation
  - \(\text{parent}(t) = s\) if \((t, k-1, q)\) is a predecessor of \((s, k, p)\)
  - In fact, this is usually done in IC3 as well (for trace reconstruction)
Recursive Blocking Stage in Quip (1)

1. Each time that we examine a proof obligation \((s, k, p)\), check whether \(s\) intersects a reachable state \(r \in R\).

2. Discover new reachable states when possible
   - Claim: if \(s\) intersects \(r \in R\) and if \(\text{parent}(s)\) exists, then there exists a reachable state \(r'\) that intersects \(\text{parent}(s)\).
     - Indeed, \textbf{ALL} states in \(s\) lead to a state in \(\text{parent}(s)\).
     - Therefore \(r\) leads to a state in \(\text{parent}(s)\) as well.
   - A similar idea is present in: C. Wu, C. Wu, C. Lai, C. Huang: \textit{A counterexample-guided interpolant generation algorithm for SAT-based model checking}. TCAD 2014.

3. When \((s, k, p)\) is blocked by an inductive lemma \(\neg t\), add \((t, k+1, \text{may})\) as a new proof obligation
   - Push \(\neg t\) to \(F_{k+1}\) instead of blocking \((s, k+1)\).

4. Clear all proof obligations if their number becomes too large (important, not in pseudocode).
Recursive Blocking Stage in Quip (2)

// Find a reachable state \( r \in s \), or strengthen the inductive trace
s.t. \( F_N \Rightarrow \neg s \)

Quip_recBlockCube(\( s, N, q \))
Add(\( Q, (s, N, q) \))

while \( \neg \text{Empty}(Q) \) do
  \( (s, k, p) \leftarrow \text{Pop}(Q) \)
  if \( (k = 0) \) && \( (p = \text{must}) \) return “Counterexample”
  if \( (k = 0) \) && \( (p = \text{may}) \)
    find a state \( r \) one-step-reachable from \( \text{Init} \),
    such that \( r \) intersects parent(s)
    add \( r \) to \( R \); continue
  if \( (F_k \Rightarrow \neg s) \) continue
  if \( (s \) intersects some state \( r \in R) \) && \( (p = \text{must}) \) return
    “Counterexample”
  if \( (s \) intersects some state \( r \in R) \) && \( (p = \text{may}) \)
    if parent(s) exists, find a state \( r' \) one-step-reachable
    from \( r \),
    such that \( r' \) intersects parent(s)
    add \( r' \) to \( R \); continue

// -- continued on the next slide --
Recursive Blocking Stage in Quip (3)

Quip_recBlockCube(s, N, p)
// -- continued from the previous slide --
if (F_{k-1} \land Tr \land s') is SAT
    t ← generalized predecessor of s
    Add(Q, (t, k-1, p))
    Add(Q, (s, k, p))
else
    \neg t ← generalize \neg s by inductive
    generalization (to level m \geq k)
    add \neg t to F_m
if (m < N)
    if (t = s) Add(Q, (t, m+1, p))
else    Add(Q, (t, m+1, may))
    // attempt to block t (not s)
Experiments: IC3 vs. Quip on HWMCC’13 and ’14

<table>
<thead>
<tr>
<th></th>
<th>UNSAFE solved</th>
<th>UNSAFE time</th>
<th>SAFE solved</th>
<th>SAFE time</th>
</tr>
</thead>
<tbody>
<tr>
<td>IC3</td>
<td>22 (2)</td>
<td>52,302</td>
<td>76 (7)</td>
<td>137,244</td>
</tr>
<tr>
<td>Quip</td>
<td>32 (12)</td>
<td>20,302</td>
<td>99 (30)</td>
<td>69,590</td>
</tr>
</tbody>
</table>

Experimental results on the instances solved by either IC3 or Quip separated into unsafe and safe instances. The numbers in parentheses represent the unique solves. The times are in seconds.

- Implemented in IBM formal verification tool *Rulebase-Sixthsense*
- Data for 140 instances that were not trivially solved by preprocessing but could be solved either by IC3 or Quip within 1-hour
- Detailed results at http://arieg.bitbucket.org/quip
Experiments: IC3 vs. Quip on HWMCC’13 and ‘14

- Data for 140 instances from prev slide
Quip – future work

- Improve handling of forward reachable states (both for performance and memory)
- Generalize forward reachable states
- Incorporate these ideas with other known IC3 developments
  - Abstraction-Refinement:
    Y. Vizel, O. Grumberg, S. Shoham: *Lazy abstraction and SAT-based reachability in hardware model checking*. FMCAD 2012
  - Lemma generalization:
    Z. Hassan, A. Bradley, F. Somenzi: *Better Generalization in IC3*. FMCAD 2013
- Experiment with other ways to combine the ideas into a full algorithm
- Lift Quip to more general domains
K-Induction without Unrolling

FMCAD’17

Arie Gurfinkel
Alexander Ivrii
K-induction Principle

Induction:

\[
P(s_0)
\]
\[
\forall i: P(s_i) \Rightarrow P(s_{i+1})
\]
\[
\forall i: P(s_i)
\]

k-step Induction:

\[
P(s_{0..k-1})
\]
\[
\forall i: P(s_{i..i+k-1}) \Rightarrow P(s_{i+k})
\]
\[
\forall i: P(s_i)
\]
SAT-based Model Checking with K-induction

Let the k-unrolling of transition relation be defined as:

\[ U_k = T^{<0>} \land T^{<1>} \land \ldots \land T^{<k-1>} \]

Use SAT solver to check validity of two formulas:

- Base case:
  \[ I^{<0>} \land U_{k-1} \Rightarrow P^{<0>} \ldots P^{<k-1>} \]

- Induction step:
  \[ U_k \land P^{<0>} \ldots P^{<k-1>} \Rightarrow P^{<k>} \]

If both are valid, then P is true in all the reachable states.

If the base case is invalid, there is a counterexample.

If the induction step is invalid, increase k and try again.
Simple path assumption

Unfortunately, k-induction is not complete

- some properties are not k-inductive for any k
- for example,

Simple path restriction:

- There is a path to \( \neg P \) iff there is a simple path to \( \neg P \) (path with no repeated states)
Complete k-induction with simple paths

Let simple(s\textsubscript{0..k}) be defined as:
- \( \forall i, j \text{ in } 0..k: (i \neq j) \Rightarrow s_i \neq s_j \)

k-induction over simple paths:

\[
P(s_{0..k-1}) \quad \forall i: \text{simple}(s_{0..k}) \wedge P(s_{i..i+k-1}) \Rightarrow P(s_{i+k})
\]

\( \forall i: P(s_i) \)

Must hold for k large enough, since a length of the longest simple path is bounded (recurrence diameter)
Terminology: k-invariants and k-induction

- \( \varphi \) is an invariant if it holds on all reachable states

- \( \varphi \) is a k-invariant if it holds on all states reachable in up to \( k \) steps:
  \[
  \text{Init}(X_0) \land \text{Tr}(X_0, X_1) \land \ldots \land \text{Tr}(X_{N-1}, X_N) \Rightarrow \varphi(X_N) \text{ for all } 0 \leq N \leq k
  \]

- \( \varphi \) is a k-inductive invariant if \( \varphi \) is a (k-1)-invariant, and
  \[
  \varphi(X_0) \land \text{Tr}(X_0, X_1) \land \ldots \land \varphi(X_{k-1}) \land \text{Tr}(X_{k-1}, X_k) \Rightarrow \varphi(X_k)
  \]

- k-induction states that if \( \varphi \) is a k-inductive invariant, then \( \varphi \) is an invariant
k-induction vs IC3

• **k-induction** and **IC3** have complementary strengths, both theoretically and practically

• Can we understand both algorithms in a common way?
  • we present an IC3-like algorithm to show whether a given safety property is k-inductive

• Can we devise an effective algorithm combining the two?
  • we show how IC3 can be extended with k-inductive reasoning – with minor modifications of the IC3-algorithm
k-induction vs IC3

k-induction and IC3 algorithms are heavily used for unbounded model checking in both hardware and software domains.

Theoretically, the two algorithms have complementary strengths:
• k-induction (with loop-free constraints) is stronger than 1-induction
• IC3 derives new lemmas to strengthen the property

Practically (on hardware benchmarks)
• k-induction is mostly successful for small values of k (up to 10)
• IC3 solves many more properties than k-induction
• However, there are properties that can be proved by k-induction with low value of k, but IC3 “gets lost”
k-induction without unrolling

• We present K-IND – an algorithm to decide whether a (k-1)-invariant formula $\varphi$ is k-inductive

• K-IND returns:
  • $\varphi$ is k-inductive, OR
  • $\varphi$ is not k-inductive (and a counterexample to k-induction)

• Highlights:
  • Does not unroll the transition relation

  • Guarantees loop-free constraints without introducing expensive unique-state constraints
K-IND: for k=3 and without loop-free constraints

• Let $\varphi$ be 2-invariant
  • No Init-state can reach a $\neg \varphi$-state in 0, 1 or 2 steps
  • Taking 0, 1 or 2 successive predecessors of a $\neg \varphi$-state cannot get to an Init-state

• We want to check/determine whether $\varphi$ is 3-inductive invariant
  • $\varphi(X_0) \land Tr(X_0, X_1) \land \varphi(X_1) \land Tr(X_1, X_2) \land \varphi(X_2) \land Tr(X_2, X_3) \Rightarrow \varphi(X_3)$?

• Equivalently, we want to check whether the following formula is satisfiable
  • $\varphi(X_0) \land Tr(X_0, X_1) \land \varphi(X_1) \land Tr(X_1, X_2) \land \varphi(X_2) \land Tr(X_2, X_3) \land \neg \varphi(X_3)$

• Equivalently, $\varphi$ is a 3-inductive invariant if and only if we cannot start from a $\neg \varphi$-state and find 3 successive predecessors satisfying $\varphi$
K-IND: demonstration

- We use an IC3-like algorithm to check satisfiability:
  \[ \varphi(X_0) \land Tr(X_0, X_1) \land \varphi(X_1) \land Tr(X_1, X_2) \land \varphi(X_2) \land Tr(X_2, X_3) \land \neg \varphi(X_3) \]

- For simplicity, assume that \( \varphi = \neg s \), where \( s \) is a cube over registers

- \( t \) is a predecessor of \( s \)
- \( u \) is a predecessor of \( t \)
- \( v \) is a predecessor of \( u \)
- We have found 3 successive predecessors: \( \varphi \) is not 3-inductive
  - Moreover, if \( v \) is an Init-state, then we have a counter-example to \( \varphi \)
K-IND: demonstration

• We use an IC3-like algorithm to check satisfiability:
  \[ \varphi(X_0) \land \text{Tr}(X_0, X_1) \land \varphi(X_1) \land \text{Tr}(X_1, X_2) \land \varphi(X_2) \land \text{Tr}(X_2, X_3) \land \neg \varphi(X_3) \]

• For simplicity, assume that \( \varphi = \neg s \), where \( s \) is a cube over registers

\[
\varphi \land \text{Tr} \land t' \quad \varphi \land \text{Tr} \land s'
\]

• Suppose now that \( t \) has no predecessors satisfying \( \varphi \)

• As in IC3, we can learn a lemma \( \neg t \) that explains why \( t \) is not reachable
  • All SAT queries are made relative to the same frame
  • All lemmas “hold for all frames”
  • No need re-enqueue discharged proof obligations
K-IND: demonstration

• Putting it all together:
  • Keep a stack of proof obligations \{ (s, i) \}, initially \{ (\neg \varphi, 0) \}
    • Where i represents the “depth” rather than “level”
  • Keep a set of lemmas G, initially empty
  • Iteratively:
    • Take the top proof obligation t and make a predecessor query
      \[ G \land \varphi \land Tr \land t' \]
    • If found a depth-3 predecessor u and u is Init, return “Counter-Example”
    • Else if found a depth-3 predecessor, return “Counter-Example to 2-induction”
    • Else if found a predecessor u, add a new proof obligation (u, i-1)
    • Else adds \neg t to G
    • If G \Rightarrow \neg s, return “Blocked”

• As in IC3, we can generalize \neg t as long as Init \Rightarrow \neg t and G \land \varphi \land Tr \land \neg t \Rightarrow \neg t'
  • in this case, the algorithm may return “Blocked” even when \varphi is not 3-inductive
  • But, G is still an inductive invariant proving \varphi
K-IND: loop-free constraints

• To integrate simple-path constraints, add (the negations of) all parent states to the predecessor queries

\[
\neg u \land \neg t \land \neg s \land \varphi \land \text{Tr} \land u' \\
\neg t \land \neg s \land \varphi \land \text{Tr} \land t' \\
\neg s \land \varphi \land \text{Tr} \land s' \]

• Here s, t, u, v are generalized proof obligations (= sets of states)
K-IND: relatively k-inductive

- Let $F$ be 0-invariant ($\text{Init} \Rightarrow F$)

- $\varphi$ is a k-inductive invariant relative to $F$ if $\varphi$ is a k-invariant, and

\[
\varphi(X_0) \land F(X_0) \land \text{Tr}(X_0, X_1) \land \ldots \land \varphi(X_{k-1}) \land F(X_{k-1}) \land \text{Tr}(X_{k-1}, X_k) \Rightarrow \varphi(X_k)
\]

- K-IND can be extended to determine if $\varphi$ is k-inductive relative to $F$
K-IND: experimental results

In practice (on hardware benchmarks, and without loop-free constraints):

• K-IND solves a few more properties than k-induction
  • Due to lemma generalization

• When solved by both, k-induction is usually faster than K-IND
  • Exactly in the same way as BMC is usually faster than IC3 when looking for counterexamples
KIC3: K-Inductive IC3

• Related Work: PD-KIND
  • “Property-Directed k-Induction”, Dejan Jovanović and Bruno Dutertre, FMCAD’2016
  • Variant of IC3/PDR based on k-induction
  • Effective on SMT benchmarks
  • Requires unrolling transition relation for validating k-inductive queries
    • A direct implementation of PD-KIND for hardware does not scale
  • Has strongly inspired our solution

• We present KIC3
  • A framework extending IC3 with k-inductive reasoning
  • Integrates k-induction into IC3 with minor modifications of the IC3-framework
  • But does not fully incorporate loop-free constraints
**k-inductive blocking**

- Given a proof obligation \((s, i)\), we can attempt to block \(s\) using k-induction relative to \(F_{i-1}\)
  - Can choose any \(k \leq i\)

- Let’s suppose that \(F_{i-1} \land \neg s \land Tr \land s’\) is satisfiable, and \(t\) is a predecessor of \(s\) in \(F_{i-1}\)

- IC3:
  - Adds a new proof obligation \((t, i-1)\)
  - Proceeds to checking whether \(F_{i-2} \land \neg t \land Tr \land t’\) is satisfiable

- k-inductive blocking (for \(k > 1\)):
  - Adds a new proof obligation \((t, i)\)
  - Proceeds to checking whether \(F_{i-1} \land \neg t \land Tr \land t’\) is satisfiable
Comparison of KIC3-blocking and IC3-blocking

• What happens if both \( F_{i-2} \land \neg t \land Tr \land t' \) and \( F_{i-1} \land \neg t \land Tr \land t' \) are UNSAT?
  • IC3 learns a lemma valid (at least) to \( F_{i-1} \)
  • k-inductive blocking learns a lemma valid (at least) to \( F_i \)
  • Lemmas learned by k-inductive blocking are in general of “higher quality”

• What happens if \( F_{i-2} \land \neg t \land Tr \land t' \) is UNSAT, while \( F_{i-1} \land \neg t \land Tr \land t' \) is SAT?
  • IC3 stops blocking \((t, i-1)\) and returns to blocking \((s, i)\)
  • k-inductive blocking continues blocking predecessors of \( t \)
  • If k-inductive blocking succeeds blocking the topmost proof-obligation \((s, i)\)
    • It may learn more lemmas, but all of these lemmas are valid (at least) to \( F_i \)

• Recall that using k-induction to block \((s, i)\) may return “counterexample-to-k-induction”
  • Simple solution: fall back to blocking \((s, i)\) using IC3
  • Inspired by PD-KIND: block this counterexample-to-k-induction using IC3* and continue blocking \((s, i)\) using k-induction
    • *or we can again use m-inductive blocking for a suitable value of \( m \)
Alternative ways to think about KIC3-blocking

- k-inductive blocking is IC3 with a slightly different strategy for managing proof obligations
  - IC3 schedules proof obligations at the lowest level they are unknown
  - k-inductive blocking may schedule proof obligations to higher levels instead

- k-inductive blocking can be thought of as abstraction
  - Given a proof obligation \((t, j)\), IC3 checks whether \(\neg t\) is inductive relative to \(F_{j-1}\)
  - However, any abstraction of \(F_{j-1}\) can be used instead
  - For example, using only lemmas from \(F_{i-1}\) closely corresponds to KIC3
Which states to block using KIC3-blocking?

• For the experiments, we modified the procedure for recursive blocking of \((\neg \text{Bad}, i)\):
  • First, use k-inductive blocking of \((\neg \text{Bad}, i)\)
  • If unsuccessful, block \((\neg \text{Bad}, i)\) as usual

• Can also use k-inductive blocking during the pushing stage of the IC3 algorithm
  • Directly inspired by PD-KIND
Experimental Results

- Implemented in IBM’s formal verification tool on top of Quip

- 238 single-property designs from HWMCC’15 (all the designs that are not solved by simple logic synthesis, but are solved either by at least one configuration considered)

- Experimented with $k =$ the induction depth, and $m =$ the number of counterexamples to k-induction blocked using IC3

- 15-minutes time-limit (per property)
Experimental Results

• Increasing k, while fixing m=0:
  • Slightly in favor of using k-induction
  • Runtimes are highly correlated

• The scatter plot has k=5 and m=0
  • Points above diagonal = wins for IC3
  • Points below diagonal = wins for KIC3

  • IC3 solves 230 properties in 52,776 s
  • KIC3 solves 233 properties in 51,695 s

• Similar observations on proprietary designs and larger time-limits
Experimental Results

- Increasing \( m \), while fixing \( k \):
  - Significantly degrades performance
  - Runtimes are less correlated

- The scatter plot has \( k=5 \) and \( m=5 \)
  - Points above diagonal = wins for IC3
  - Points below diagonal = wins for KIC3

- IC3 solves 230 properties in 52,776 s
- KIC3 solves 224 properties in 57,864 s

- Similar observations on proprietary designs and larger time-limits
Conclusion

IC3 is a great algorithm for hardware Model Checking
  • but, it can still be improved

QUIP: QUest for an Inductive Proof
  • aggressively push existing lemmas
  • enlarge initial state by computing reachable states
  • use reachable states to prune bad lemmas

KIC3: IC3 and k-induction
  • k-induction without unrolling (and without simple path constraints)
  • integrates easily into IC3 framework
  • expensive, hard to control when to apply

IC3 is a great framework to explore MC strategies