The Science, Art and Magic of Constrained Horn Clauses

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https://notebooks.azure.com/arie-gurfinkel/projects/spacerexamples

Software Model Checking of Programs / Transitions Systems / Push-down Systems

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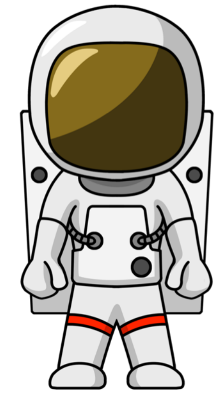
Satisfiability of Constrained Horn Logic (CHC) fragment of First Order Logic



Reduce Model Checking to FOL Satisfiability

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Constrained Horn Clauses (CHC)

Alfred Horn

Is it short for Horner?

Is it related to hornets?

Is it Santa Clause blowing a Horn?

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Example CHC: Is this SAT?

$$\forall x \cdot x \leq 0 \implies P(x)$$
$$\forall x, x' \cdot P(x) \wedge x < 5 \wedge x' = x + 1 \implies P(x')$$
$$\forall x \cdot P(x) \wedge x \geq 10 \implies \mathit{false}$$

This set of clauses is satisfiable

The model is an extension of the standard model of arithmetic with:

$$P(x) \equiv \{x \mid x \leq 5\}$$
$$\equiv \{5, 4, 3, 2, \ldots\}$$

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Constrained Horn Clauses (CHC)

A Constrained Horn Clause (CHC) is a FOL formula of the form

$$\forall V \cdot (\varphi \wedge p_1[X_1] \wedge \cdots \wedge p_n[X_n]) \rightarrow h[X]$$

where

- φ - constraint in a background theory $\mathcal{T}$
- $\mathcal{T}$ - background theory
  - Linear Arithmetic, Arrays, Bit-Vectors, or combinations
- V - variables, and X_i are terms over V
- $p_1, \ldots, p_n, h$ - n-ary predicates
- $p_i[X]$ - application of a predicate to first-order terms

CHC Satisfiability

Π - set of CHCs

M - $\mathcal{T}$-**model** of a set of Π

- M satisfies $\mathcal{T}$
- M satisfies Π – through first-order interpretation of each predicate p_i

A set of clauses is **satisfiable** if and only if it has a model

- This is the usual FOL satisfiability

$\mathcal{T}$-solution of a set of CHCs Π is a substitution σ from predicates p_i to $\mathcal{T}$-formulas such that $\Pi\sigma$ is $\mathcal{T}$-valid

In the context of program verification

Program $\models \boldsymbol{\varphi}$	**iff**	$CHC_{Program} \rightarrow \varphi$
Inductive Invariant	=	Solution to CHC
Counter Example Trace	=	Resolution proof of CHC

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The model is an extension of the standard model of arithmetic with:

$$\begin{aligned} P(x) &\equiv \{x \mid x \leq 5\} \\ &\equiv \{5, 4, 3, 2, \ldots\} \end{aligned}$$

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Validating the solution

Original CHC

$$\forall x \cdot x \leq 0 \implies P(x)$$

$$\forall x, x' \cdot P(x) \wedge x < 5 \wedge x' = x + 1 \implies P(x')$$

$$\forall x \cdot P(x) \wedge x \geq 10 \implies \mathit{false}$$

Validation of P(x) = {x | x <= 5}

$$\vdash \forall x \cdot x \leq 0 \implies x \leq 5$$

$$\vdash \forall x, x' \cdot x \leq 5 \wedge x < 5 \wedge x' = x + 1 \implies x' \leq 5$$

$$\vdash \forall x \cdot x \leq 5 \wedge x \geq 10 \implies \mathit{false}$$

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Example CHC: is this SAT?

$$\forall x \cdot x \leq 0 \implies Q(x)$$
$$\forall x, x' \cdot Q(x) \land x < 5 \land x' = x + 1 \implies Q(x')$$
$$\forall x \cdot Q(x) \land x \geq 2 \implies \mathit{false}$$

This set of clauses is unsatisfiable

Justification is a refutation by resolution and instantiation

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Example CHC: is this SAT?

$$\forall x \cdot x \leq 0 \implies Q(x)$$
$$\forall x, x' \cdot Q(x) \wedge x < 5 \wedge x' = x + 1 \implies Q(x')$$
$$\forall x \cdot Q(x) \wedge x \geq 2 \implies false$$

Refutation

$$\cfrac{(x=0)\ \cfrac{\forall x \cdot x \leq 0 \implies Q(x)}{Q(0)} \qquad \forall x \cdot Q(x) \wedge x < 5 \implies Q(x+1)}{\cfrac{Q(1) \quad \forall x \cdot Q(x) \wedge x < 5 \implies Q(x+1)}{\cfrac{Q(2) \quad \forall x \cdot Q(x) \wedge x \geq 2 \implies false}{false}}}$$

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Horn Clauses for Program Verification

$\ell_{out}(x_0, w', e_o)$, which is an entry point into successor edges.
with the edges are formulated as follows:

$$p_{init}(x_0, \boldsymbol{w}, \bot) \leftarrow x = x_0 \quad \text{where } x \text{ occurs in } \boldsymbol{w}$$
$$p_{exit}(x_0, ret, \top) \leftarrow \ell(x_0, \boldsymbol{w}, \top) \quad \text{for each label } \ell \text{, and re}$$
$$p(x, ret, \bot, \bot) \leftarrow p_{exit}(x, ret, \bot)$$
$$p(x, ret, \bot, \top) \leftarrow p_{exit}(x, ret, \top)$$
$$\ell_{out}(x_0, \boldsymbol{w}', e_o) \leftarrow \ell_{in}(x_0, \boldsymbol{w}, e_i) \wedge \neg e_i \wedge \neg wlp(S, \neg(e_i =$$

```
5. incorrect :- Z=W+1, W≥0, W+1<
                read(A,W,U), read(A,Z
6. p(I1,N,B) :- 1≤I, I<N, D=I−1, I1=I+1, V=U+1,
                read(A,D,U), write(A
7. p(I,N,A) :- I=1, N>1.
```

De Angelis et al. Verifying Array Programs by Transforming Verification Conditions. VMCAI'14

Weakest Preconditions If we apply Boogie directly we obtain a translation from programs to Horn logic using a weakest liberal pre-condition calculus [26]:

$$\mathsf{ToHorn}(program) := wlp(Main(), \top) \wedge \bigwedge_{decl \in program} \mathsf{ToHorn}(decl)$$
$$\mathsf{ToHorn}(\mathbf{def}\ p(x)\ \{S\}) := wlp\begin{pmatrix}\mathbf{havoc}\ x_0; \mathbf{assume}\ x_0 = x; & \\ \mathbf{assume}\ p_{pre}(x); S, & p(x_0, ret)\end{pmatrix}$$
$$wlp(x := E, Q) := \mathbf{let}\ x = E\ \mathbf{in}\ Q$$
$$wlp((\mathbf{if}\ E\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2), Q) := wlp(((\mathbf{assume}\ E; S_1)\square(\mathbf{assume}\ \neg E; S_2)), Q)$$
$$wlp((S_1 \square S_2), Q) := wlp(S_1, Q) \wedge wlp(S_2, Q)$$
$$wlp(S_1; S_2, Q) := wlp(S_1, wlp(S_2, Q))$$
$$wlp(\mathbf{havoc}\ x, Q) := \forall x\ .\ Q$$
$$wlp(\mathbf{assert}\ \varphi, Q) := \varphi \wedge Q$$
$$wlp(\mathbf{assume}\ \varphi, Q) := \varphi \rightarrow Q$$
$$wlp((\mathbf{while}\ E\ \mathbf{do}\ S), Q) := inv(\boldsymbol{w}) \wedge \forall \boldsymbol{w}\ .\ \begin{pmatrix}((inv(\boldsymbol{w}) \wedge E) \rightarrow wlp(S, inv(\boldsymbol{w}))) \\ \wedge\ ((inv(\boldsymbol{w}) \wedge \neg E) \rightarrow Q)\end{pmatrix}$$

To translate a procedure call $\ell : y := q(E); \ell'$ within a procedure p, create he clauses:

$$p(\boldsymbol{w}_0, \boldsymbol{w}_4) \leftarrow p(\boldsymbol{w}_0, \boldsymbol{w}_1), call(\boldsymbol{w}_1, \boldsymbol{w}_2), q(\boldsymbol{w}_2, \boldsymbol{w}_3), return(\boldsymbol{w}_1, \boldsymbol{w}_3, \boldsymbol{w}_4)$$
$$q(\boldsymbol{w}_2, \boldsymbol{w}_2) \leftarrow p(\boldsymbol{w}_0, \boldsymbol{w}_1), call(\boldsymbol{w}_1, \boldsymbol{w}_2)$$
$$call(\boldsymbol{w}, \boldsymbol{w}') \leftarrow \pi = \ell, x' = E, \pi' = \ell_{q_{init}}$$
$$return(\boldsymbol{w}, \boldsymbol{w}', \boldsymbol{w}'') \leftarrow \pi' = \ell_{q_{exit}}, \boldsymbol{w}'' = \boldsymbol{w}[ret'/y, \ell'/\pi]$$

Bjørner, Gurfinkel, McMillan, and Rybalchenko:

Horn Clause Solvers for Program Verification

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Horn Clauses for Concurrent / Distributed / Parameterized Systems

For assertions $R_1, \ldots, R_N$ over V and $E_1, \ldots, E_N$ over V, V',

$$
\begin{array}{lll}
\text{CM1}: & \mathit{init}(V) & \rightarrow R_i(V) \\
\text{CM2}: & R_i(V) \wedge \rho_i(V, V') & \rightarrow R_i(V') \\
\text{CM3}: & (\bigvee_{i \in 1..N \setminus \{j\}} R_i(V) \wedge \rho_i(V, V')) & \rightarrow E_j(V, V') \\
\text{CM4}: & R_i(V) \wedge E_i(V, V') \wedge \rho_i^{=}(V, V') & \rightarrow R_i(V') \\
\text{CM5}: & R_1(V) \wedge \cdots \wedge R_N(V) \wedge \mathit{error}(V) & \rightarrow \mathit{false}
\end{array}
$$

multi-threaded program P is safe

Rybalchenko et al. Synthesizing Software Verifiers from Proof Rules. PLDI'12

$$\left\{ R(\mathsf{g}, \mathsf{p}_{\sigma(1)}, \mathsf{l}_{\sigma(1)}, \ldots, \mathsf{p}_{\sigma(k)}, \mathsf{l}_{\sigma(k)}) \leftarrow \mathit{dist}(\mathsf{p}_1, \ldots, \mathsf{p}_k) \wedge R(\mathsf{g}, \mathsf{p}_1, \mathsf{l}_1, \ldots, \mathsf{p}_k, \mathsf{l}_k) \right\}_{\sigma \in S_k} \quad (6)$$

$$R(\mathsf{g}, \mathsf{p}_1, \mathsf{l}_1, \ldots, \mathsf{p}_k, \mathsf{l}_k) \leftarrow \mathit{dist}(\mathsf{p}_1, \ldots, \mathsf{p}_k) \wedge \mathit{Init}(\mathsf{g}, \mathsf{l}_1) \wedge \cdots \wedge \mathit{Init}(\mathsf{g}, \mathsf{l}_k) \quad (7)$$

$$R(\mathsf{g}', \mathsf{p}_1, \mathsf{l}'_1, \ldots, \mathsf{p}_k, \mathsf{l}_k) \leftarrow \mathit{dist}(\mathsf{p}_1, \ldots, \mathsf{p}_k) \wedge \left((\mathsf{g}, \mathsf{l}_1) \xrightarrow{\mathsf{p}_1} (\mathsf{g}', \mathsf{l}'_1)\right) \wedge R(\mathsf{g}, \mathsf{p}_1, \mathsf{l}_1, \ldots, \mathsf{p}_k, \mathsf{l}_k) \quad (8)$$

$$R(\mathsf{g}', \mathsf{p}_1, \mathsf{l}_1, \ldots, \mathsf{p}_k, \mathsf{l}_k) \leftarrow \mathit{dist}(\mathsf{p}_0, \mathsf{p}_1, \ldots, \mathsf{p}_k) \wedge \left((\mathsf{g}, \mathsf{l}_0) \xrightarrow{\mathsf{p}_0} (\mathsf{g}', \mathsf{l}'_0)\right) \wedge \mathit{RConj}(0, \ldots, k) \quad (9)$$

$$\mathit{false} \leftarrow \mathit{dist}(\mathsf{p}_1, \ldots, \mathsf{p}_r) \wedge \left(\bigwedge_{j=1,\ldots,m} (\mathsf{p}_j = p_j \wedge (\mathsf{g}, \mathsf{l}_j) \in E_j) \right) \wedge \mathit{RConj}(1, \ldots, r) \quad (10)$$

Figure 4: Horn constraints encoding a homogeneous infinite system with the help of a k-indexed invariant. S_k is the symmetric group on $\{1, \ldots, k\}$, i.e., the group of all permutations of k numbers; as an optimisation, any generating subset of S_k, for instance transpositions, can be used instead of S_k. In (10), we define $r = \max\{m, k\}$.

Hojjat et al. Horn Clauses for Communicating Timed Systems. HCVS'14

$$Init(i, j, \overline{v}) \wedge Init(j, i, \overline{v}) \wedge$$
$$Init(i, i, \overline{v}) \wedge Init(j, j, \overline{v}) \Rightarrow I_2(i, j, \overline{v})$$
$$I_2(i, j, \overline{v}) \wedge Tr(i, \overline{v}, \overline{v}') \Rightarrow I_2(i, j, \overline{v}') \quad (3)$$
$$I_2(i, j, \overline{v}) \wedge Tr(j, \overline{v}, \overline{v}') \Rightarrow I_2(i, j, \overline{v}') \quad (4)$$
$$I_2(i, j, \overline{v}) \wedge I_2(i, k, \overline{v}) \wedge I_2(j, k, \overline{v}) \wedge$$
$$Tr(k, \overline{v}, \overline{v}') \wedge k \neq i \wedge k \neq j \Rightarrow I_2(i, j, \overline{v}') \quad (5)$$
$$I_2(i, j, \overline{v}) \Rightarrow \neg Bad(i, j, \overline{v})$$

Figure 3: $VC_2(T)$ for two-quantifier invariants.

Gurfinkel et al. SMT-Based Verification of Parameterized Systems. FSE 2016

(initial) $\quad \mathsf{init}(g, x_1) \wedge \cdots \wedge \mathsf{init}(g, x_n) \rightarrow Inv(g, \ell_{\mathsf{init}}, x_1, \ldots, \ell_{\mathsf{init}}, x_k)$

(inductive) $\quad Inv(g, \ell_1, x_1, \ldots, \ell_i, x_i, \ldots, \ell_k, x_k) \wedge s(g, x_i, g', x'_i) \rightarrow Inv(g', \ell_1, x_1, \ldots, \ell'_i, x'_i, \ldots, \ell_k,$

(non-interference) $\quad Inv(g, \ell_1, x_1, \ldots, \ell_k, x_k) \wedge$
$Inv(g, \ell^\dagger, x^\dagger, \ell_2, x_2, \ldots, \ell_k, x_k) \wedge$
$\vdots$
$Inv(g, \ell_1, x_1, \ldots, \ell_{k-1}, x_{k-1}, \ell^\dagger, x^\dagger) \wedge s(g, x^\dagger, g', \cdot) \rightarrow Inv(g', \ell_1, x_1, \ldots, \ell_k, x_k)$

(safe) $\quad Inv(g, \ell_1, x_1, \ldots, \ell_k, x_k) \wedge \mathsf{err}(g, \ell_1, x_1, \ldots, \ell_m, x_m) \rightarrow \mathit{false}$

Figure 6. Horn clause encoding for thread modularity at level k (where (ℓ_i, s, ℓ'_i) and $(\ell^\dagger, s, \cdot)$ refer to statement s on a
from ℓ_i to ℓ'_i and, respectively, from $\ell^\dagger$ to some other location in the control flow graph)

Hoenicke et al. Thread Modularity at Many Levels. POPL'17

Program Verification with HORN(LIA)

```
z = x; i = 0;
assume (y > 0);
while (i < y) {
  z = z + 1;
  i = i + 1;
}
assert(z == x + y);
```

IS SAT?

```
z = x & i = 0 & y > 0                        →  Inv(x, y, z, i)
Inv(x, y, z, i) & i < y & z1=z+1 & i1=i+1    →  Inv(x, y, z1, i1)
Inv(x, y, z, i) & i >= y & z != x+y          →  false
```

In SMT-LIB

```
(set-logic HORN)

;; Inv(x, y, z, i)
(declare-fun Inv ( Int Int Int Int) Bool)

(assert
 (forall ( (A Int) (B Int) (C Int) (D Int))
         (=> (and (> B 0) (= C A) (= D 0))
            (Inv A B C D)))
 )
(assert
 (forall ( (A Int) (B Int) (C Int) (D Int) (C1 Int) (D1 Int) )
         (=>
          (and (Inv A B C D) (< D B) (= C1 (+ C 1)) (= D1 (+ D
1)))
          (Inv A B C1 D1)
          )
         )
 )
(assert
 (forall ( (A Int) (B Int) (C Int) (D Int))
         (=> (and (Inv A B C D) (>= D B) (not (= C (+ A B))))
            false
            )
         )
 )

(check-sat)
(get-model)
```

```
$ z3 add-by-one.smt2
sat
(model
  (define-fun Inv ((x!0 Int) (x!1 Int) (x!2 Int) (x!3 Int)) Bool
    (and (<= (+ x!2 (* (- 1) x!0) (* (- 1) x!3)) 0)
         (<= (+ x!2 (* (- 1) x!0) (* (- 1) x!1)) 0)
         (<= (+ x!0 x!3 (* (- 1) x!2)) 0)))
)
```

```
Inv(x, y, z, i)
   z  = x + i
   z <= x + y
```

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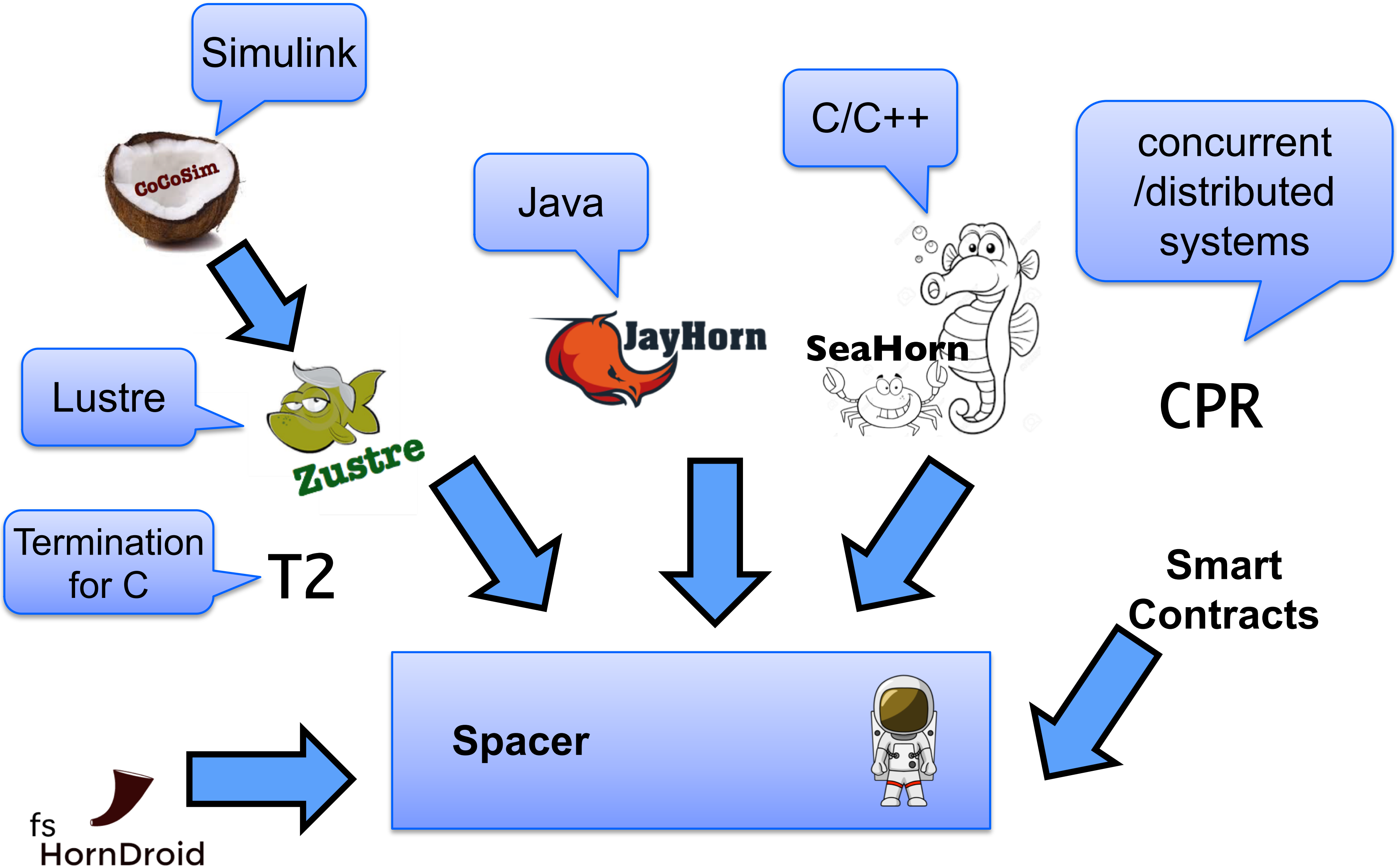
Logic-based Algorithmic Verification



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INTERACTIVE TUTORIAL

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Procedures for Solving CHC(T)

Predicate abstraction by lifting Model Checking to HORN

- QARMC, Eldarica, …

Maximal Inductive Subset from a finite Candidate space (Houdini)

- TACAS'18: hoice, FreqHorn

Machine Learning

- PLDI'18: sample, ML to guess predicates, DT to guess combinations

Abstract Interpretation (Poly, intervals, boxes, arrays…)

- Approximate least model by an abstract domain (SeaHorn, …)

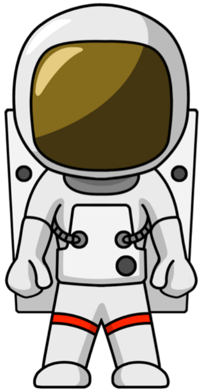
Interpolation-based Model Checking

- Duality, QARMC, …

SMT-based Unbounded Model Checking (IC3/PDR)

- Spacer, Implicit Predicate Abstraction

Spacer: Solving SMT-constrained CHC



Spacer: SAT procedure for SMT-constrained Horn Clauses

- now the default CHC solver in Z3
  - https://github.com/Z3Prover/z3
  - dev branch at https://github.com/agurfinkel/z3

Supported SMT-Theories

- Linear Real and Integer Arithmetic
- Quantifier-free theory of arrays
- Universally quantified theory of arrays + arithmetic
- Best-effort support for many other SMT-theories
  - data-structures, bit-vectors, non-linear arithmetic

Support for Non-Linear CHC

- for procedure summaries in inter-procedural verification conditions
- for compositional reasoning: abstraction, assume-guarantee, thread modular, etc.

A little bit of complexity

Satisfiability of CHC over most interesting theories is undecidable

- e.g., CHC(Linear Real Arithmetic), CHC(Linear Integer Arithmetic)
- proof: many easy reductions, for example, counter automata

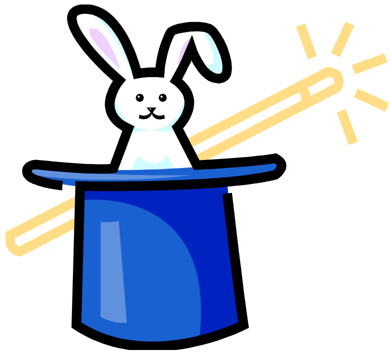
Satisfiability of Linear CHC over Propositional logic is decidable

- Finite state model checking of transition systems
- Complexity: linear in the size of the graph induced by the transition system

Satisfiability of Non-Linear CHC over Propositional logic is decidable

- Finite state model checking of pushdown systems
- Complexity: cubic in the size of the pushdown system

Decidability of some classes of CHC: Difference arithmetic (= timed automata)



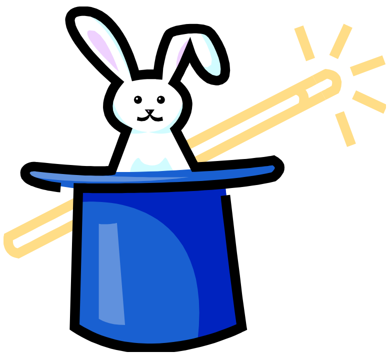
SOLVING CONSTRAINED HORN CLAUSES

A Magician's Guide to Solving Undecidable Problems

Develop a procedure ***P*** for a decidable problem

Show that ***P*** is a decision procedure for the problem

- e.g., model checking of finite-state systems



Choose one of

- Always terminate with some answer (over-approximation)
- Always make useful progress (under-approximation)

Extend procedure ***P*** to procedure ***Q*** that "solves" the undecidable problem

- Ensure that ***Q*** is still a decision procedure whenever ***P*** is
- Ensure that ***Q*** either always terminates or makes progress

Linear CHC Satisfiability

Satisfiability of a set of linear CHCs is reducible to satisfiability of THREE clauses of the form

$$Init(X) \rightarrow P(X)$$
$$P(X) \wedge Tr(X, X') \rightarrow P(X')$$
$$P(X) \rightarrow \neg Bad(X)$$

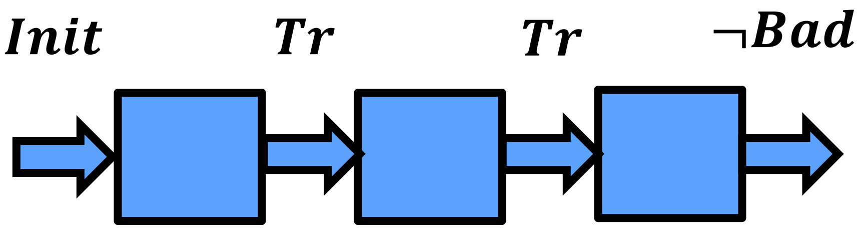
where, X' = {x' | x $\in$ X}, P a fresh predicate, and *Init*, *Bad*, and *Tr* are constraints

Proof:

add extra arguments to distinguish between predicates

$$\frac{Q(y) \wedge \phi \rightarrow W(y, z)}{P(id='Q', y) \wedge \phi \rightarrow P(id='W', y, z)}$$

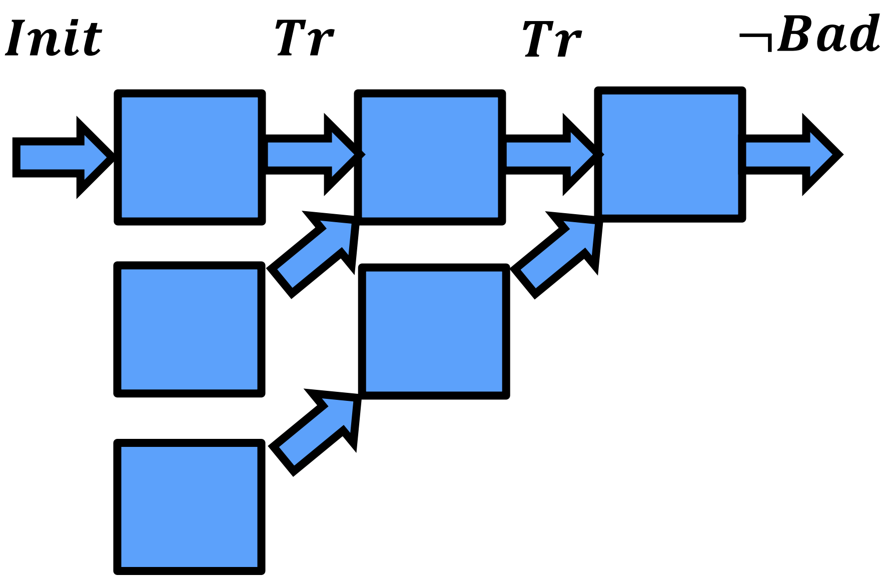
IC3, PDR and friends



Finite State Machines

(HW model checking)

[Bradley, VMCAI 2011]

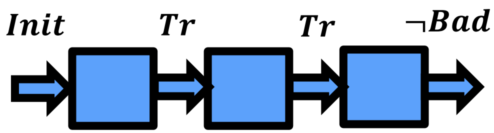


Push Down Machines

(SW model checking)

[Hoder&B, SAT 2012]

IC3, PDR and friends



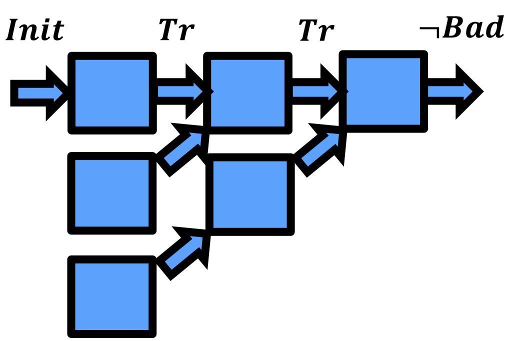
Finite State Machines

(HW model checking)

[Bradley, VMCAI 2011]

Finite State

- Incremental SAT solving [Bradley, VMCAI 11]
- Fast prime implicants [Een& FMCAD 11]
- Basis for predicate abstraction [Cimatti& TACAS 14, Birgmeier& CAV 14]



Push Down Machines

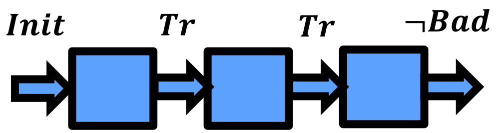
(SW model checking)

[Hoder, B, SAT 2012]

Infinite State

- Arithmetic + Farkas [H&B, SAT 12]
- Arithmetic + Model Based Projection [K&, CAV 14]
- Polyhedra + Convex Closure [B&G, VMCAI 15]
- Arithmetic + Arrays [K&, FMCAD 15]
- ∃∀ - EPR fragment [K'&, CAV 15]
- ∃∀ + Arithmetic/Arrays [G&, ATVA 18]

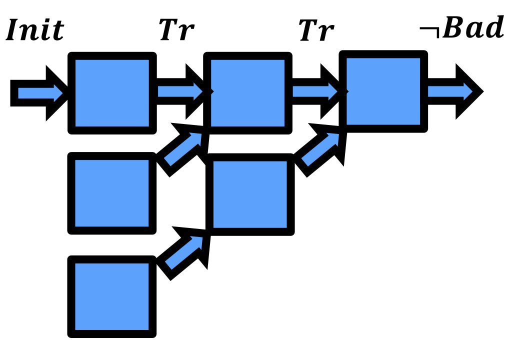
IC3, PDR and friends



Finite State Machines

(HW model checking)

[Bradley, VMCAI 2011]



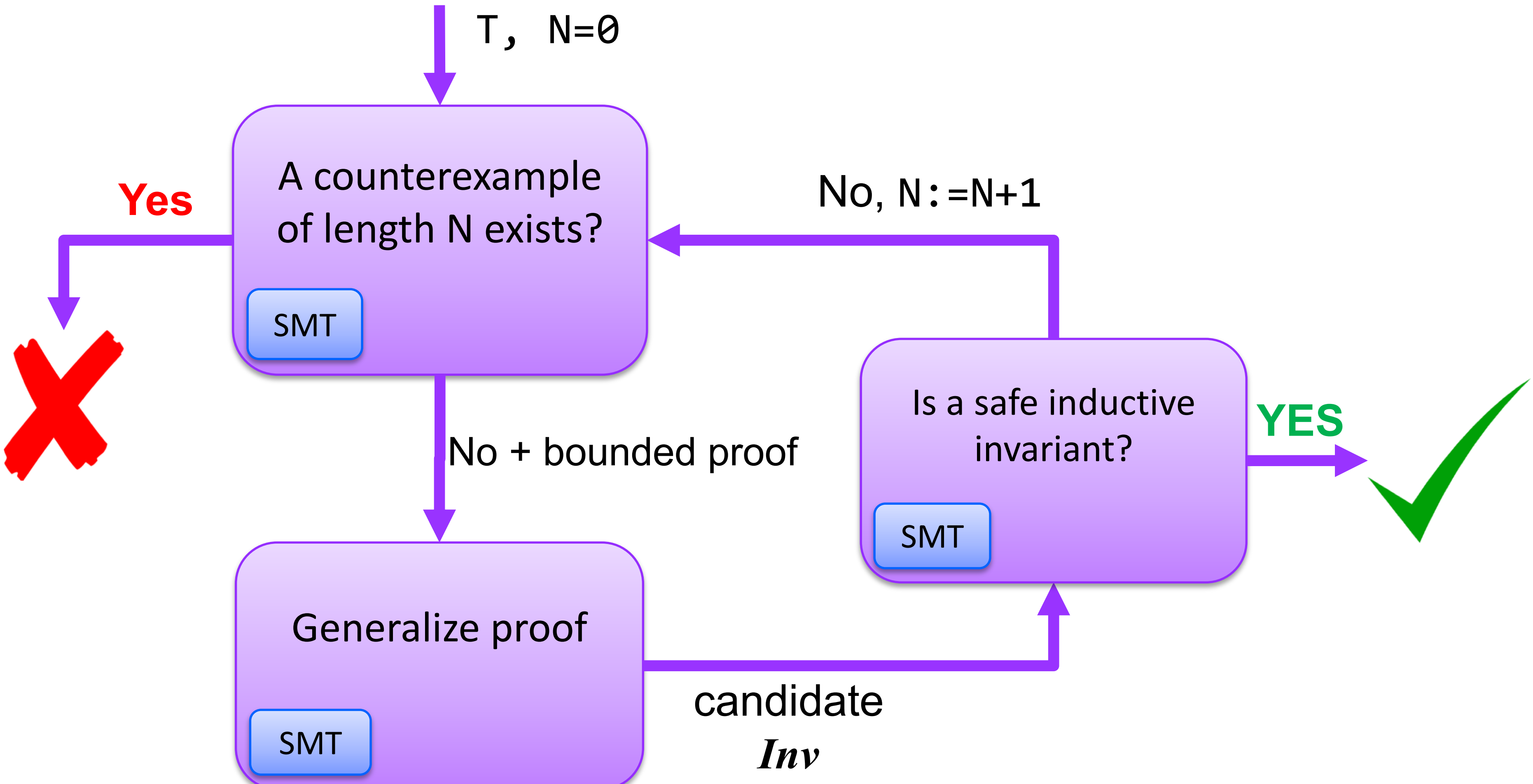
Push Down Machines

(SW model checking)

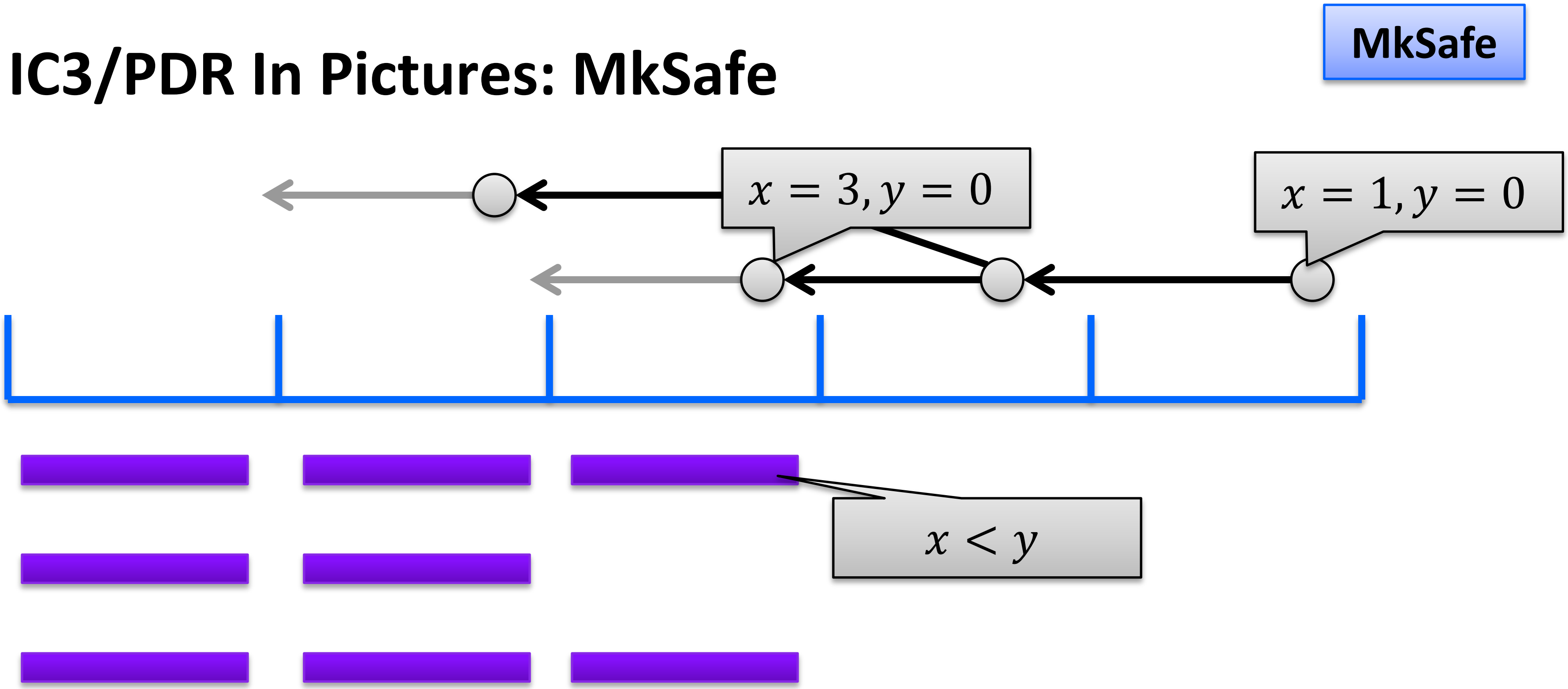
[Hoder, B, SAT 2012]

Finite State	Search Strategies
SAT	[Bradley, VMCAI 11] CTI – Counter Examples To Induction
Infinite State	[G&Ivrii, FMCAD 15] Under and over-approximations
SMT Arithmetic Arrays Quantifiers	[Vizel&G, CAV 14] Use SAT for blocking IC3 for pushing

Verification by Incremental Generalization



IC3/PDR In Pictures: MkSafe



Predecessor

$$\text{find } M \text{ s.t. } M \models F_i \wedge Tr \wedge m'$$

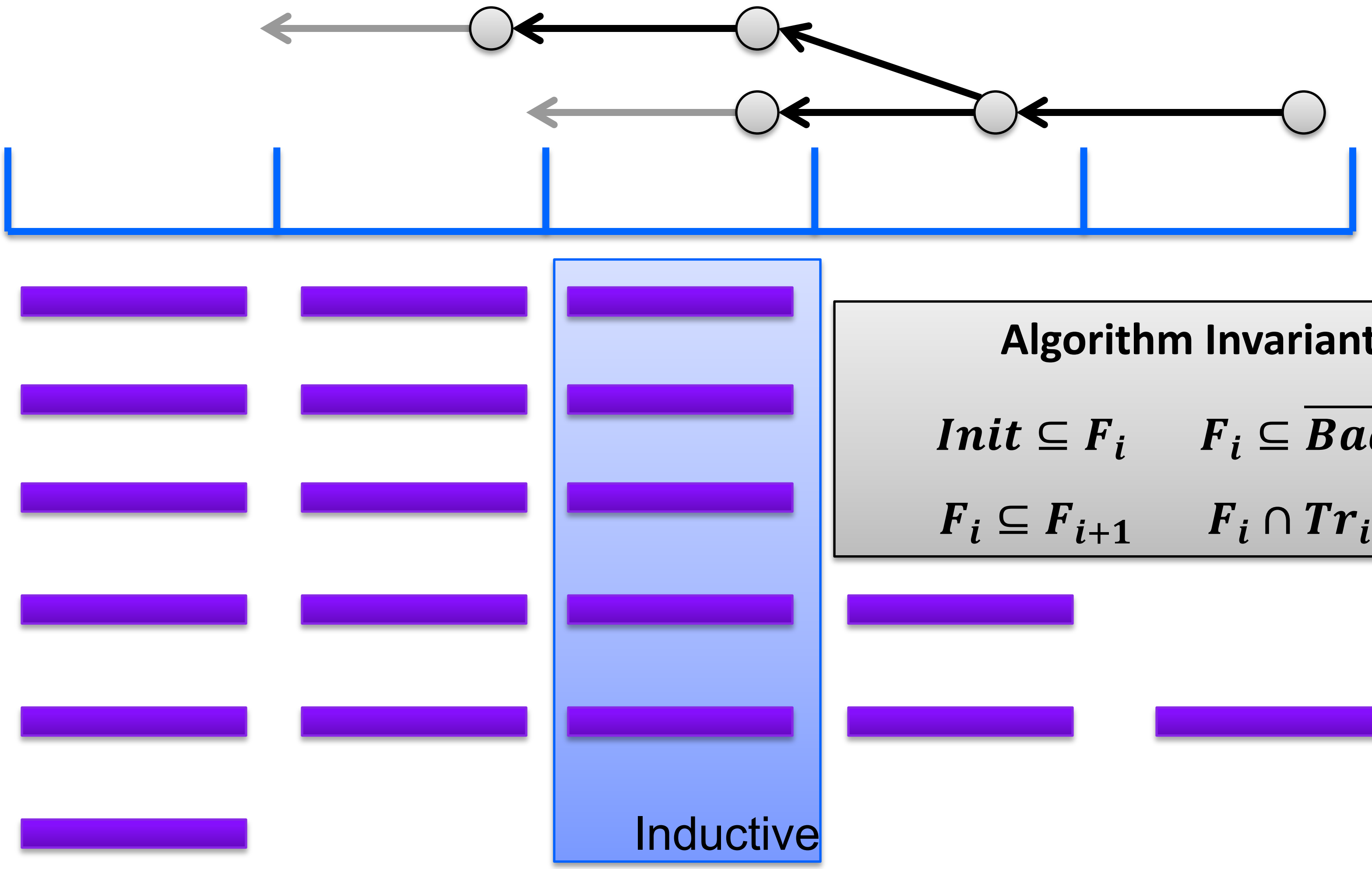
$$\text{find } m \text{ s.t. } (M \models m) \wedge (m \implies \exists V' \cdot Tr \wedge m')$$

NewLemma

$$\text{find } \ell \text{ s.t. } (F_i \wedge Tr \implies \ell') \wedge (\ell \implies \neg m)$$

IC3/PDR in Pictures: Push

Push



Algorithm Invariants

$Init \subseteq F_i \qquad F_i \subseteq \overline{Bad}$

$F_i \subseteq F_{i+1} \qquad F_i \cap Tr_i \subseteq F'_{i+1}$

SMT-query: $\vdash \ell \land F_i \land Tr \implies \ell'$

IC3/PDR: Solving Linear (Propositional) CHC

Unreachable and Reachable

- terminate the algorithm when a solution is found

Unfold

- increase search bound by 1

Candidate

- choose a bad state in the last frame

Decide

- extend a cex (backward) consistent with the current frame
- choose an assignment ***s*** s.t. $(s \wedge Fi \wedge Tr \wedge cex')$ is SAT

Conflict

- construct a lemma to explain why cex cannot be extended
- Find a clause ***L*** s.t. $L \Rightarrow \neg cex$, $Init \Rightarrow L$, and $F_i \wedge Tr \Rightarrow L'$

Induction

- propagate a lemma as far into the future as possible
- (optionally) strengthen by dropping literals

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From Propositional PDR to Solving CHC

Theories with infinitely many models

- infinitely many satisfying assignments
- can’t simply enumerate (when computing predecessor)
- can’t block one assignment at a time (when blocking)

Non-Linear Horn Clauses

- multiple predecessors (when computing predecessors)

The problem is undecidable in general, but we want an algorithm that makes progress

- doesn’t get stuck in a decidable sub-problem
- guaranteed to find a counterexample (if it exists)

IC3/PDR: Solving Linear (Propositional) CHC

Unreachable and Reachable

- terminate the algorithm when a solution is found

Unfold

- increase search bound by 1

Candidate

- choose a bad state in the last frame

Decide

- extend a cex (backward) consistent with the current frame
- choose an assignment ***s*** s.t. $(s \land Fi \land Tr \land cex')$ is SAT

Conflict

- construct a lemma to explain why cex cannot be extended
- Find a clause ***L*** s.t. $L \Rightarrow \neg cex$, $Init \Rightarrow L$, and $F_i \land Tr \Rightarrow L'$

Theory dependent

Induction

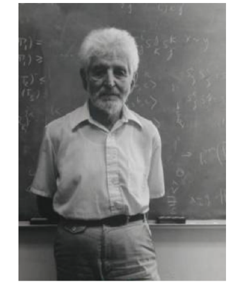
- propagate a lemma as far into the future as possible
- (optionally) strengthen by dropping literals

$$\left((\boldsymbol{F_i} \wedge \boldsymbol{Tr}) \vee \boldsymbol{Init'}\right) \Rightarrow \boldsymbol{\varphi'}, \qquad \boldsymbol{\varphi'} \Rightarrow \neg \boldsymbol{cex'}$$

Looking for φ’

CONFLICT (ARITHMETIC)

Craig Interpolation Theorem



Theorem (Craig 1957)
Let A and B be two First Order (FO) formulae such that $A \Rightarrow \neg B$, then there exists a FO formula I, denoted ITP(A, B), such that

$$A \Rightarrow I \qquad I \Rightarrow \neg B \qquad \Sigma(I) \in \Sigma(A) \cap \Sigma(B)$$

A Craig interpolant ITP(A, B) can be effectively constructed from a resolution proof of unsatisfiability of $A \wedge B$

In Model Checking, Craig Interpolation Theorem is used to safely over-approximate the set of (finitely) reachable states

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Examples of Craig Interpolation for Theories

Boolean logic

$$A = (\neg b \wedge (\neg a \vee b \vee c) \wedge a) \qquad B = (\neg a \vee \neg c)$$

$$ITP(A, B) = a \wedge c$$

Equality with Uniterpreted Functions (EUF)

$$A = (f(a) = b \wedge p(f(a))) \qquad B = (b = c \wedge \neg p(c))$$

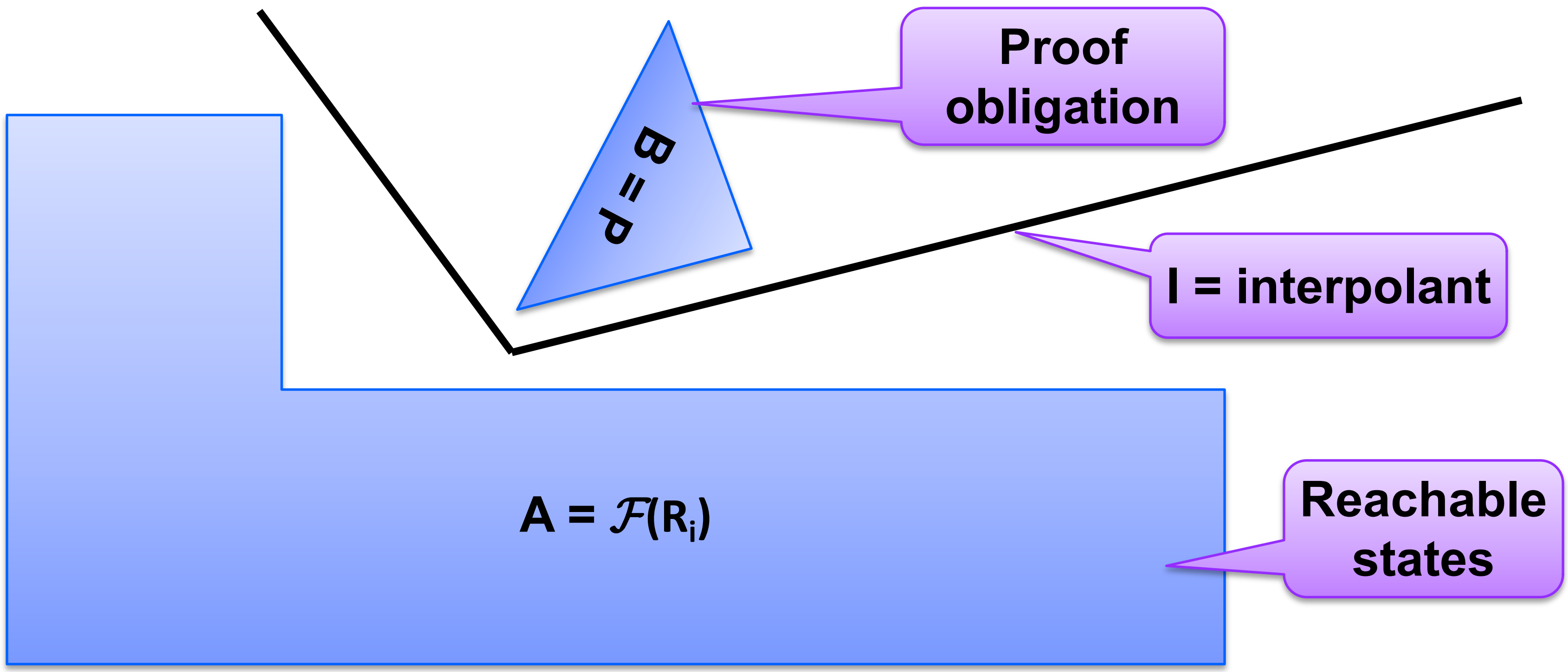
$$ITP(A, B) = p(b)$$

Linear Real Arithmetic (LRA)

$$A = (z + x + y > 10 \wedge z < 5) \qquad B = (x < -5 \wedge y < -3)$$

$$ITP(A, B) = x + y > 5$$

Craig Interpolation for Linear Arithmetic



Useful properties of existing interpolation algorithms [CGS10] [HB12]

- $I \in ITP(A, B)$ then $\neg I \in ITP(B, A)$
- if A is syntactically convex (a monomial), then I is convex
- if B is syntactically convex, then I is co-convex (a clause)
- if A and B are syntactically convex, then I is a half-space

Arithmetic Conflict

Notation: $\mathcal{F}(A) = (A(X) \wedge Tr) \vee Init(X')$.

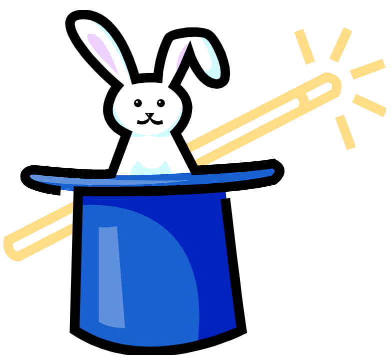
Conflict For $0 \leq i < N$, given a counterexample $\langle P, i+1 \rangle \in Q$ s.t. $\mathcal{F}(F_i) \wedge P'$ is unsatisfiable, add $P^{\uparrow} = \textsc{Itp}(\mathcal{F}(F_i), P')$ to F_j for $j \leq i+1$.

Counterexample is blocked using Craig Interpolation

- summarizes the reason why the counterexample cannot be extended

Generalization is not inductive

- weaker than IC3/PDR
- inductive generalization for arithmetic is still an open problem



Computing Interpolants for IC3/PDR

Much simpler than general interpolation problem for $A \wedge B$

- B is always a conjunction of literals
- A is dynamically split into DNF by the SMT solver
- DPLL(T) proofs do not introduce new literals

Interpolation algorithm is reduced to analyzing all theory lemmas in a DPLL(T) proof produced by the solver

- every theory-lemma that mixes B-pure literals with other literals is interpolated to produce a single literal in the final solution
- interpolation is restricted to clauses of the form $(\bigwedge B_i \Rightarrow \bigvee A_j)$

Interpolating (UNSAT) Cores

- improve interpolation algorithms and definitions to the specific case of PDR
- classical interpolation focuses on eliminating non-shared literals
- in PDR, the focus is on finding good generalizations

Farkas Lemma

Let $\Phi = t_1 \geq b_1 \wedge \ldots \wedge t_n \geq b_n$, t_i are linear terms and b_i are constants

Φ is *unsatisfiable* iff $0 \geq 1$ is derivable from Φ by resolution

- $x + 2y > 10,$
- $-x > 5,$
- $-y > 3$
- $0 = (x + 2y - x - 2y) > (10 + 5 + 2 \cdot 3) > 21$

Proof uses *Farkas* coefficients $g_1, \ldots, g_n$ such that

- $g_i > 0$
- $g_1 \cdot t_1 + \ldots + g_n \cdot t_n = 0$
- $g_1 \cdot b_1 + \ldots + g_n \cdot bn > 1$

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Frakas Lemma Example

Interpolants

$$z + x + y > 10 \qquad \times 1$$
$$-z > -5 \qquad \times 1$$

$$\Bigg\} \quad x + y > 5$$

$$-x > 5 \qquad \times 1$$
$$-y > 3 \qquad \times 1$$

$$\Bigg\} \quad x + y < -8$$

$$0 > 13$$

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Interpolation for Linear Real Arithmetic

Let A ∧ B be UNSAT, where

- $A = t_1 \geq b_1 \wedge \ldots \wedge t_i \geq bi$, and
- $B = t_{i+1} \geq bi \wedge \ldots \wedge t_n \geq b_n$

Let $g_1, \ldots, g_n$ be the Farkas coefficients witnessing UNSAT

Then

- $g_1 \cdot (t_1 - b_1) + \ldots + g_i \cdot (t_i - b_i) \geq 0$ is an interpolant between A and B
- $g_{i+1} \cdot (t_{i+1} - b_{i+1}) + \ldots + g_n \cdot (t_n - b_n) \geq 0$ is an interpolant between B and A

Program Verification with HORN(LIA)

```
z = x; i = 0;
assume (y > 0);
while (i < y) {
  z = z + 1;
  i = i + 1;
}
assert(z == x + y);
```

IS SAT?

```
z = x & i = 0 & y > 0                                  ➔  Inv(x, y, z, i)
Inv(x, y, z, i) & i < y & z1=z+1 & i1=i+1  ➔  Inv(x, y, z1, i1)
Inv(x, y, z, i) & i >= y & z != x+y            ➔  false
```

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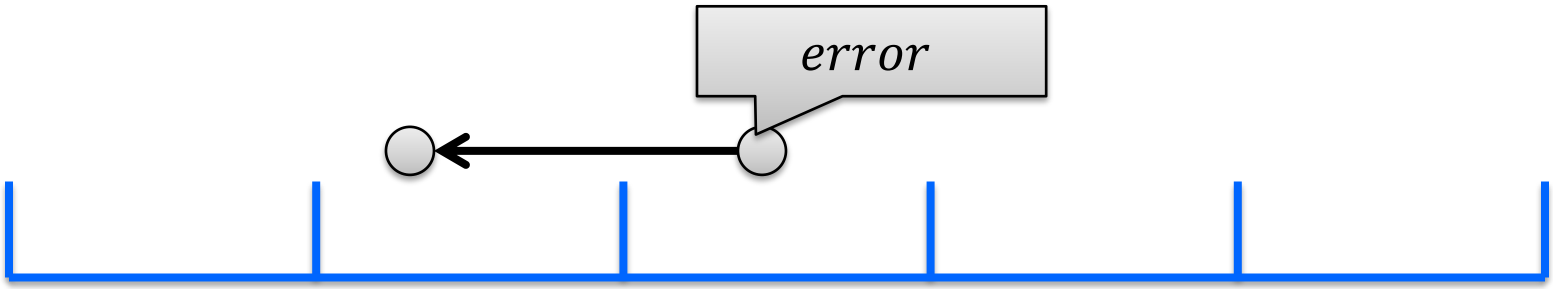
Lemma Generation Example

MkSafe



Transition Relation

$x = x_0 \wedge z = z_0+1 \wedge i=i_0+1 \wedge y > i_0$

Pob

$i >= y \wedge x + y > z$

Farkas explanation for unsat

$$\frac{\dfrac{x_0 + y_0 <= z_0,\ \ x <= x_0,\ z_0 < z,\ i <= i_0 + 1}{x + i <= z} \qquad \dfrac{i >= y,\ x+y > z}{x + i > z}}{\text{false}}$$

Learn lemma: $x + i <= z$

Interpolation Problem in Spacer

Given an arbitrary LRA formula A and a conjunction of literals s such that $A \wedge s$ are UNSAT, compute an interpolant I such that

- $s \Rightarrow I$ $\quad$ $I \wedge A \Rightarrow FALSE$ $\quad$ I is over symbols common to s and A

Use an SMT solver to decide that $s \wedge A$ are UNSAT

- SMT solver uses LRA theory lemmas (called Farkas Theory Lemmas) of the form:

  $\neg ((s_1 \wedge \ldots \wedge s_k) \wedge (a_1 \wedge \ldots \wedge a_m))$

  where s_i are literals from s and a_i are literals from A
- For each such lemma L_j, $((s_1 \wedge \ldots \wedge s_k) \wedge (a_1 \wedge \ldots \wedge a_m)$ is UNSAT
- Let t_j be an interpolant corresponding to L_j

Then, an interpolant between s and A is a clause of the form

$(\neg t_1 \vee \ldots \vee \neg t_k)$ with one literal per each theory lemma

- in practice, interpolation is optimized by examining and restructuring SMT resolution proof, dealing with Boolean reasoning, and global optimization

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Computing Interpolants in Spacer

Much simpler than general interpolation problem for $A \wedge B$

- B is always a conjunction of literals
- A is dynamically split into DNF by the SMT solver
- DPLL(T) proofs do not introduce new literals

Interpolation algorithm is reduced to analyzing all theory lemmas in a DPLL(T) proof produced by the solver

- every theory-lemma that mixes B-pure literals with other literals is interpolated to produce a single literal in the final solution
- interpolation is restricted to clauses of the form $(\bigwedge B_i \Rightarrow \bigvee A_j)$

Interpolating (UNSAT) Cores

- improve interpolation algorithms and definitions to the specific case of PDR
- classical interpolation focuses on eliminating non-shared literals
- in PDR, the focus is on finding good generalizations

$$\boldsymbol{s \subseteq pre(cex)}$$

$$\equiv$$

$$\boldsymbol{s \Rightarrow \exists X'.\, Tr(X, X') \land cex(X')}$$

Computing a predecessor ***s*** of a counterexample ***cex***

DECIDE (ARITHMETIC)

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Model Based Projection

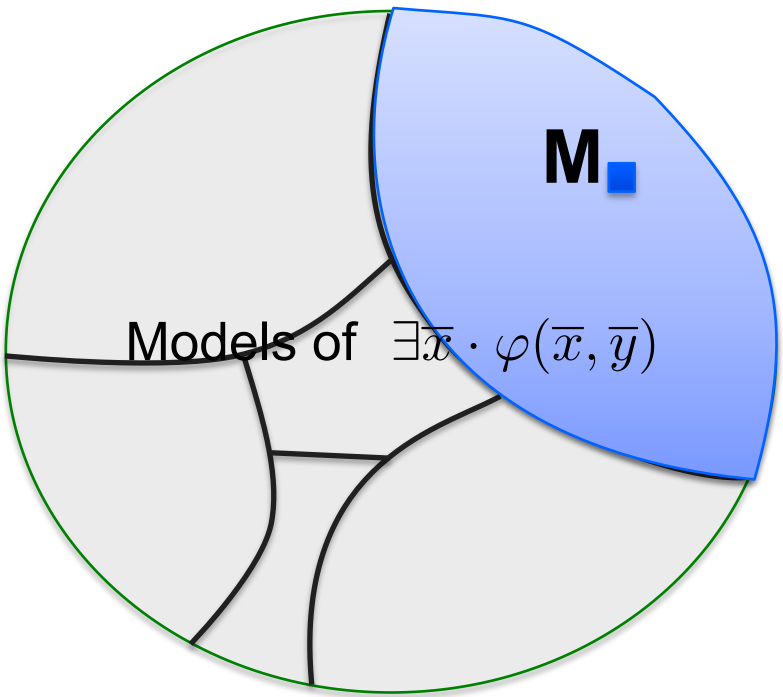
Definition: Let φ be a formula, X a set of variables, and M a model of φ. Then $\psi = MBP\ (X, M, \varphi)$ is a Model Based Projection of X, M, φ iff

1. ψ *is a monomial*
2. $Vars(\psi) \subseteq Vars(\varphi) \setminus X$
3. $M \models \psi$
4. $\psi \Rightarrow \exists X\,.\,\varphi$

Model Based Projection under-approximates existential quantifier elimination relative to a given model (i.e., satisfying assignment)

Model Based Projection

Expensive to find a quantifier-free $\psi(\overline{y}) \equiv \exists \overline{x} \cdot \varphi(\overline{x}, \overline{y})$



1. Find model M of φ (x,y)

2. Compute a partition containing M

Quantifier Elimination

Quantifier elimination procedure:

- **Input**: formula $\exists\, x\, \psi(x)$
- **Output**: equivalent φ without existential quantifier. x is eliminated.
- $\mathrm{QELIM}(\exists\, x\, \psi(x)) = \varphi$ and $\exists\, x\, \psi(x) \iff \varphi$

Quantifier elimination in propositional logic

- $\mathrm{QELIM}(\exists\, x\, \psi(x)) = \psi(\mathrm{TRUE}) \vee \psi(\mathrm{FALSE})$

Many theories support quantifier elimination (e.g., linear arithmetic)

- but not all. No quantifier elimination for EUF,
  - e.g., $(\exists x\, f(x) \neq g(x))$ cannot be expressed without the existential quantifier

Quantifier elimination is usually expensive

- e.g., propositional QELIM is exponential in the number of variables quantified

Loos-Weispfenning Quantifier Elimination for LRA

ϕ is LRA formula in Negation Normal Form

E is set of x=t atoms, U set of x < t atoms, and L set of s < x atoms

There are no other occurrences of x in ϕ[x]

$$\exists x . \varphi[x] \equiv \varphi[\infty] \vee \bigvee_{x=t \in E} \varphi[t] \vee \bigvee_{x<t \in U} \varphi[t - \epsilon]$$

where

$$(x < t')[t - \epsilon] \equiv t \leq t' \qquad (s < x)[t - \epsilon] \equiv s < t \qquad (x = e)[t - \epsilon] \equiv \mathit{false}$$

The case of lower bounds is dual

- using –∞ and t+ϵ

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Fourier–Motzkin Quantifier Elimination for LRA

$$\exists x \cdot \bigwedge_i s_i < x \wedge \bigwedge_j x < t_j$$

$$= \quad \bigwedge_i \bigwedge_j resolve(s_i < x, x < t_j, x)$$

$$= \quad \bigwedge_i \bigwedge_j s_i < t_j$$

Quadratic increase in the formula size per each eliminated variable

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Quantifier Elimination with Assumptions

$$
\begin{aligned}
& \left(\bigwedge_{j \neq 0} t_0 \leq t_j\right) \wedge \exists x \cdot \bigwedge_i s_i < x \wedge \bigwedge_j x < t_j \\
= \quad & \left(\bigwedge_{j \neq 0} t_0 \leq t_j\right) \wedge \bigwedge_i resolve(s_i < x, x < t_0, x) \\
= \quad & \left(\bigwedge_{j \neq 0} t_0 \leq t_j\right) \wedge \bigwedge_i s_i < t_0
\end{aligned}
$$

Quantifier elimination is simplified by a choice of a minimal upper bound

- For each choice of minimal upper bound, no increase in term size
- Dually, can use largest lower bound

How to chose an the assumptions?!

- MBP == use the order chosen by the model

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MBP for Linear Rational Arithmetic

Compute a **single** disjunct from LW-QE that includes the model

- Use the Model to uniquely pick a substitution term for x

$$Mbp_x(M, x = s \wedge L) = L[x \leftarrow s]$$

$$Mbp_x(M, x \neq s \wedge L) = Mbp_x(M, s < x \wedge L) \text{ if } M(x) > M(s)$$

$$Mbp_x(M, x \neq s \wedge L) = Mbp_x(M, -s < -x \wedge L) \text{ if } M(x) < M(s)$$

$$Mbp_x(M, \bigwedge_i s_i < x \wedge \bigwedge_j x < t_j) = \bigwedge_i s_i < t_0 \wedge \bigwedge_j t_0 \leq t_j \text{ where } M(t_0) \leq M(t_i), \forall i$$

MBP techniques have been developed for

- Linear Rational Arithmetic, Linear Integer Arithmetic
- Theories of Arrays, and Recursive Data Types

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Arithmetic Decide

Notation: $\mathcal{F}(A) = (A(X) \wedge Tr(X, X') \vee Init(X')$.

Decide If $\langle P, i+1 \rangle \in Q$ and there is a model $m(X, X')$ s.t. $m \models \mathcal{F}(F_i) \wedge P'$, add $\langle P_\downarrow, i \rangle$ to Q, where $P_\downarrow = \text{MBP}(X', m, \mathcal{F}(F_i) \wedge P')$.

Compute a predecessor using Model Based Projection

To ensure progress, Decide must be finite

- finitely many possible predecessors when all other arguments are fixed

Alternatively

- Completeness can follow from an interaction of **Decide** and **Conflict**
  - but requires more rules to propagate implicants backward (as in PDR) and forward (as in Spacer and Quip)

PolyPDR: Solving CHC(LRA)

Unreachable and Reachable

- terminate the algorithm when a solution is found

Unfold

- increase search bound by 1

Candidate

- choose a bad state in the last frame

Decide

- extend a cex (backward) consistent with the current frame
- find a model **M** of ***s*** s.t. $(F_i \land Tr \land cex')$, and let **s** = MBP(X', $F_i \land Tr \land cex'$)

Conflict

- construct a lemma to explain why cex cannot be extended
- Find an interpolant ***L*** s.t. $L \Rightarrow \neg cex$, $Init \Rightarrow L$, and $F_i \land Tr \Rightarrow L'$

Induction

- propagate a lemma as far into the future as possible
- (optionally) strengthen by dropping literals

Non-Linear CHC Satisfiability

Satisfiability of a set of arbitrary (i.e., linear or non-linear) CHCs is reducible to satisfiability of THREE (3) clauses of the form

$$Init(X) \to P(X)$$
$$P(X) \land P(X^o) \land Tr(X, X^o, X') \to P(X')$$
$$P(X) \to \neg Bad(X)$$

where, $X' = \{x' \mid x \in X\}$, $X^o = \{x^o \mid x \in X\}$, P a fresh predicate, and Init, Bad, and Tr are constraints

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Generalized GPDR

Input: A safety problem $\langle Init(X), Tr(X, X^o, X'), Bad(X)\rangle$.
Output: *Unreachable* or *Reachable*
Data: A cex queue Q, where a cex $\langle c_0, \ldots, c_k\rangle \in Q$ is a tuple, each $c_j = \langle m, i\rangle$, m is a cube over state variables, and $i \in \mathbb{N}$. A level N. A trace $F_0, F_1, \ldots$
Notation: $\mathcal{F}(A, B) = Init(X') \vee (A(X) \wedge B(X^o) \wedge Tr)$, and $\mathcal{F}(A) = \mathcal{F}(A, A)$
Initially: $Q = \emptyset$, $N = 0$, $F_0 = Init$, $\forall i > 0 \cdot F_i = \emptyset$
Require: $Init \to \neg Bad$
repeat

- **Unreachable** If there is an $i < N$ s.t. $F_i \subseteq F_{i+1}$ **return** *Unreachable*.
- **Reachable** if exists $t \in Q$ s.t. for all $\langle c, i\rangle \in t$, $i = 0$, **return** *Reachable*.
- **Unfold** If $F_N \to \neg Bad$, then set $N \leftarrow N + 1$ and $Q \leftarrow \emptyset$.
- **Candidate** If for some m, $m \to F_N \wedge Bad$, then add $\langle\langle m, N\rangle\rangle$ to Q.
- **Decide** If there is a $t \in Q$, with $c = \langle m, i+1\rangle \in t$, $m_1 \to m$, $l_0 \wedge m_0^o \wedge m_1'$ is satisfiable, and $l_0 \wedge m_0^o \wedge m_1' \to F_i \wedge F_i^o \wedge Tr \wedge m'$ then add $\hat{t}$ to Q, where $\hat{t} = t$ with c replaced by two tuples $\langle l_0, i\rangle$, and $\langle m_0, i\rangle$.
- **Conflict** If there is a $t \in Q$ with $c = \langle m, i+1\rangle \in t$, s.t. $\mathcal{F}(F_i) \wedge m'$ is unsatisfiable. Then, add $\varphi = \textsc{Itp}(\mathcal{F}(F_i), m')$ to F_j, for all $0 \le j \le i+1$.
- **Leaf** If there is $t \in Q$ with $c = \langle m, i\rangle \in t$, $0 < i < N$ and $\mathcal{F}(F_{i-1}) \wedge m'$ is unsatisfiable, then add $\hat{t}$ to Q, where $\hat{t}$ is t with c replaced by $\langle m, i+1\rangle$.
- **Induction** For $0 \le i < N$ and a clause $(\varphi \vee \psi) \in F_i$, if $\varphi \notin F_{i+1}$, $\mathcal{F}(\phi \wedge F_i) \to \phi'$, then add φ to F_j, for all $j \le i+1$.

until ∞;

counterexample is a tree

two predecessors

theory-aware **Conflict**

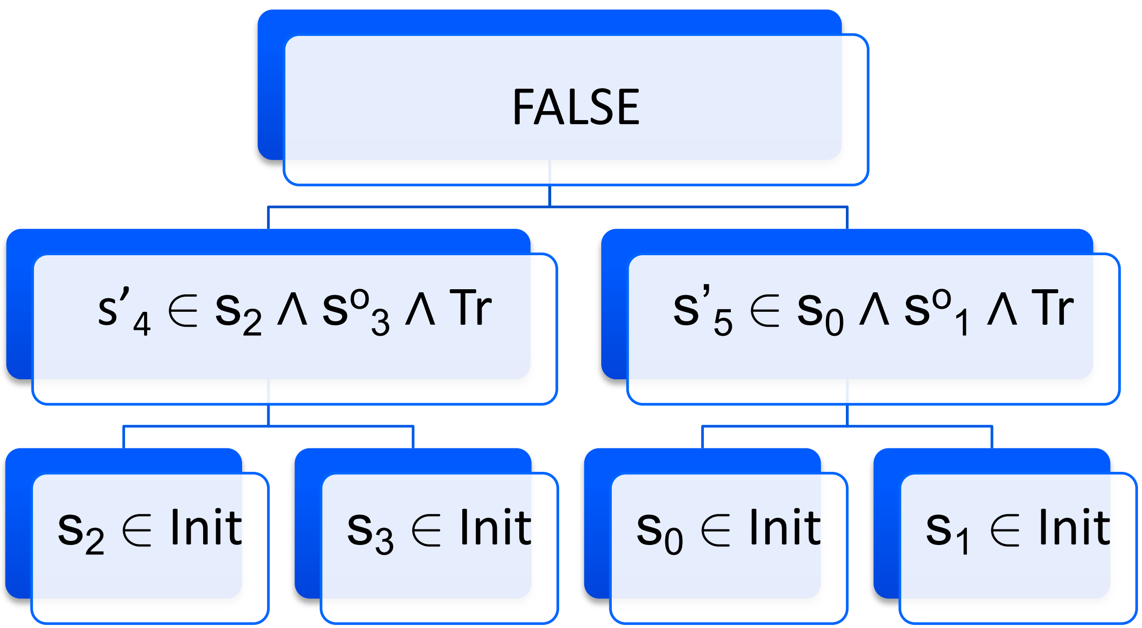
Counterexamples to non-linear CHC

A set S of CHC is unsatisfiable iff S can derive FALSE

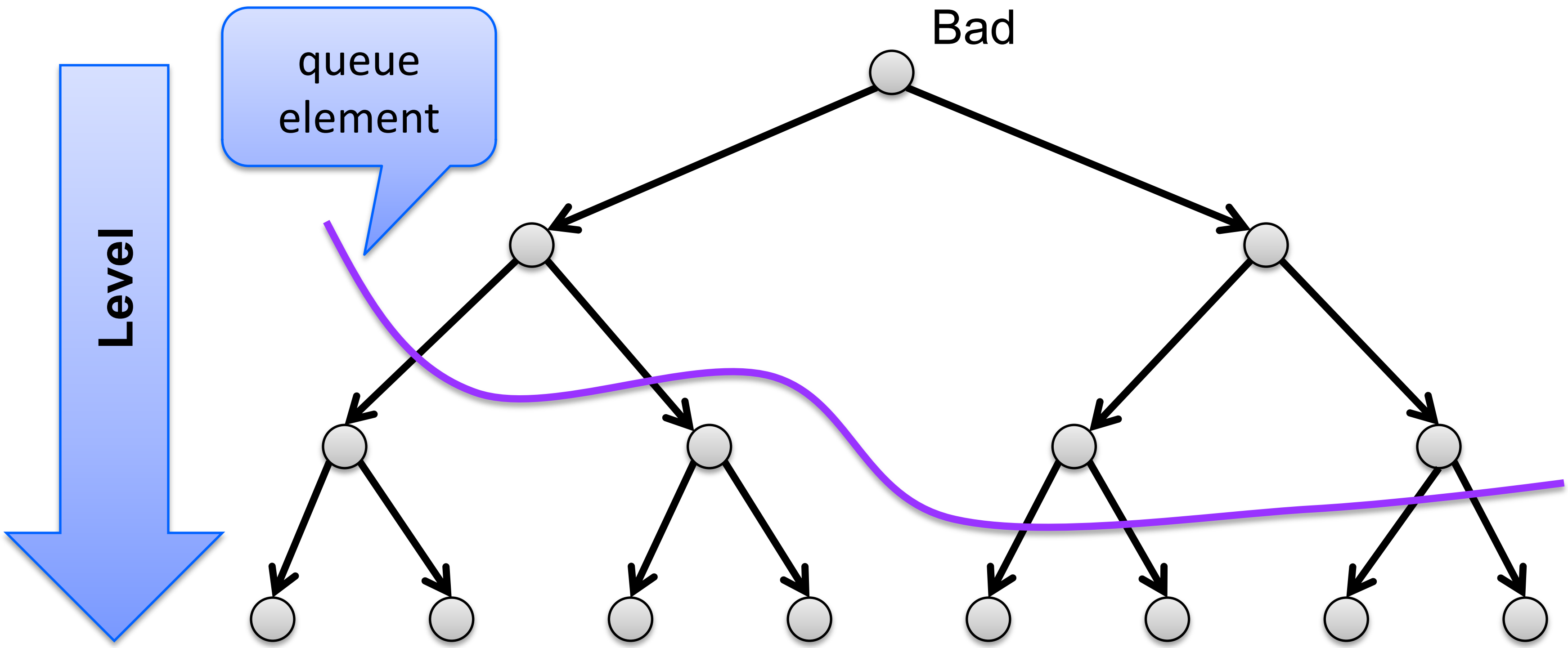
- we call such a derivation a counterexample

For linear CHC, the counterexample is a path

For non-linear CHC, the counterexample is a tree



GPDR Search Space



In Decide, one POB in the frontier is chosen and its two children are expanded

GPDR: Splitting predecessors

Consider a clause

$$P(x) \wedge P(y) \wedge x > y \wedge z = x + y \implies P(z)$$

How to compute a predecessor for a proof obligation z > 0

Predecessor over the constraint is:

$$\begin{aligned} & \exists z \cdot x > y \wedge z = x + y \wedge z > 0 \\ = \quad & x > y \wedge x + y > 0 \end{aligned}$$

Need to create two separate proof obligation

- one for P(x) and one for P(y)
- gpdr solution: split by substituting values from the model (incomplete)

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GPDR: Deciding predecessors

Decide If there is a $t \in Q$, with $c = \langle m, i+1 \rangle \in t$, $m_1 \to m$, $l_0 \wedge m_0^o \wedge m_1'$ is satisfiable, and $l_0 \wedge m_0^o \wedge m_1' \to F_i \wedge F_i^o \wedge Tr \wedge m'$ then add $\hat{t}$ to Q, where $\hat{t} = t$ with c replaced by two tuples $\langle l_0, i \rangle$, and $\langle m_0, i \rangle$.

Compute two predecessors at each application of **GPDR/Decide**

Can explore both predecessors in parallel

- e.g., BFS or DFS exploration order

Number of predecessors is unbounded

- incomplete even for finite problem (i.e., non-recursive CHC)

No caching/summarization of previous decisions

- worst-case exponential for Boolean Push-Down Systems

Spacer

Same queue as in IC3/PDR

Cache Reachable states

Three variants of **Decide**

Same **Conflict** as in APDR/GPDR

Input: A safety problem $\langle \mathit{Init}(X), \mathit{Tr}(X, X^o, X'), \mathit{Bad}(X) \rangle$.
Output: *Unreachable* or *Reachable*
Data: A cex queue Q, where a cex $c \in Q$ is a pair $\langle m, i \rangle$, m is a cube over state variables, and $i \in \mathbb{N}$. A level N. A set of reachable states $\textsc{Reach}$. A trace $F_0, F_1, \ldots$
Notation: $\mathcal{F}(A, B) = \mathit{Init}(X') \vee (A(X) \wedge B(X^o) \wedge \mathit{Tr})$, and $\mathcal{F}(A) = \mathcal{F}(A, A)$
Initially: $Q = \emptyset$, $N = 0$, $F_0 = \mathit{Init}$, $\forall i > 0 \cdot F_i = \emptyset$, $\textsc{Reach} = \mathit{Init}$
Require: $\mathit{Init} \to \neg \mathit{Bad}$
repeat

Unreachable If there is an $i < N$ s.t. $F_i \subseteq F_{i+1}$ **return** *Unreachable*.

Reachable If $\textsc{Reach} \wedge \mathit{Bad}$ is satisfiable, **return** *Reachable*.

Unfold If $F_N \to \neg \mathit{Bad}$, then set $N \leftarrow N + 1$ and $Q \leftarrow \emptyset$.

Candidate If for some m, $m \to F_N \wedge \mathit{Bad}$, then add $\langle m, N \rangle$ to Q.

Successor If there is $\langle m, i+1 \rangle \in Q$ and a model M $M \models \psi$, where $\psi = \mathcal{F}(\vee \textsc{Reach}) \wedge m'$. Then, add s to $\textsc{Reach}$, where $s' \in \text{MBP}(\{X, X^o\}, \psi)$.

DecideMust If there is $\langle m, i+1 \rangle \in Q$, and a model M $M \models \psi$, where $\psi = \mathcal{F}(F_i, \vee \textsc{Reach}) \wedge m'$. Then, add s to Q, where $s \in \text{MBP}(\{X^o, X'\}, \psi)$.

DecideMay If there is $\langle m, i+1 \rangle \in Q$ and a model M $M \models \psi$, where $\psi = \mathcal{F}(F_i) \wedge m'$. Then, add s to Q, where $s^o \in \text{MBP}(\{X, X'\}, \psi)$.

Conflict If there is an $\langle m, i+1 \rangle \in Q$, s.t. $\mathcal{F}(F_i) \wedge m'$ is unsatisfiable. Then, add $\varphi = \textsc{Itp}(\mathcal{F}(F_i), m')$ to F_j, for all $0 \leq j \leq i + 1$.

Leaf If $\langle m, i \rangle \in Q$, $0 < i < N$ and $\mathcal{F}(F_{i-1}) \wedge m'$ is unsatisfiable, then add $\langle m, i+1 \rangle$ to Q.

Induction For $0 \leq i < N$ and a clause $(\varphi \vee \psi) \in F_i$, if $\varphi \notin F_{i+1}$, $\mathcal{F}(\phi \wedge F_i) \to \phi'$, then add φ to F_j, for all $j \leq i + 1$.

until ∞;

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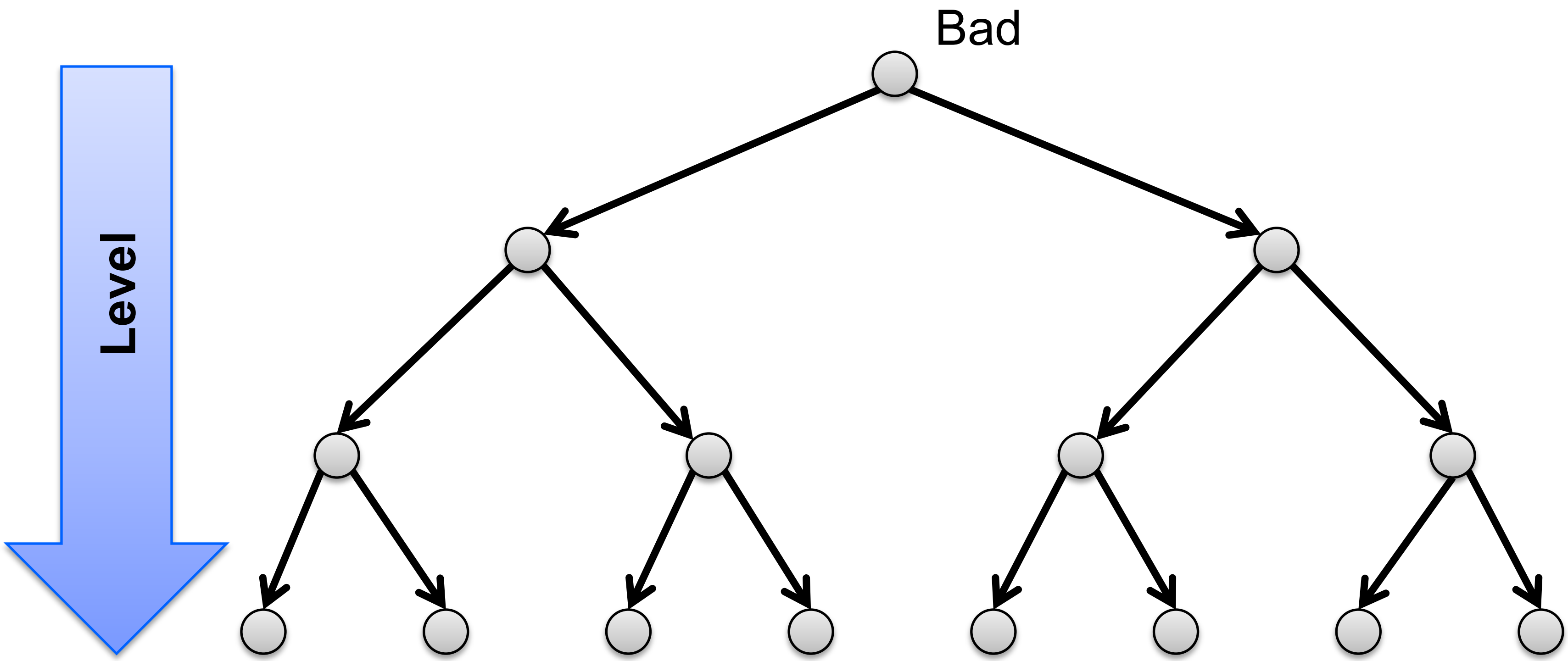
SPACER Search Space



In Decide, unfold the derivation tree in a fixed depth-first order

- use MBP to decide on counterexamples

Successor: Learn new facts (reachable states) on the way up

- use MBP to propagate facts bottom up

Successor Rule: Computing Reachable States

Successor If there is $\langle m, i+1\rangle \in Q$ and a model M $M \models \psi$, where $\psi = \mathcal{F}(\vee \textsc{Reach}) \wedge m'$. Then, add s to $\textsc{Reach}$, where $s' \in \text{MBP}(\{X, X^o\}, \psi)$.

Computing new reachable states by under-approximating forward image using MBP

- since MBP is finite, guarantee to exhaust all reachable states

Second use of MBP

- orthogonal to the use of MBP in Decide
- can allow REACH to contain auxiliary variables, but this might explode

For Boolean CHC, the number of reachable states is bounded

- complexity is polynomial in the number of states
- same as reachability in Push Down Systems

Decide Rule: Must and May refinement

DecideMust If there is $\langle m, i+1 \rangle \in Q$, and a model M $M \models \psi$, where $\psi = \mathcal{F}(F_i, \vee \textsc{Reach}) \wedge m'$. Then, add s to Q, where $s \in \text{MBP}(\{X^o, X'\}, \psi)$.

DecideMay If there is $\langle m, i+1 \rangle \in Q$ and a model M $M \models \psi$, where $\psi = \mathcal{F}(F_i) \wedge m'$. Then, add s to Q, where $s^o \in \text{MBP}(\{X, X'\}, \psi)$.

DecideMust

- use computed summary (REACH) to skip over a call site

DecideMay

- use over-approximation of a calling context to guess an approximation of the call-site
- the call-site either refutes the approximation (**Conflict**) or refines it with a witness (**Successor**)

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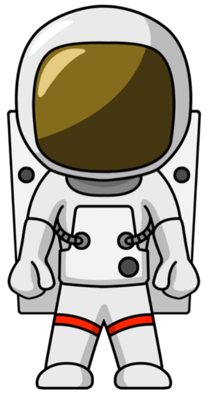
Art, Science, and Magic



Verification of Safety Properties is FOL satisfiability

- Logic: Constrained Horn Clauses (CHC)
- “Decision” procedure: Spacer
- Now with (universal) quantifiers!

Art: finding the right encoding from the problem domain to logic

- the difference between easy to impossible
- encodings can “simulate” specialized algorithms

Science: Progress, termination (when decidable)

- while the underlying problem is undecidable, many fragment or sub-problems are decidable

Magic: actually solving useful problems

- interpolation, heuristics, generalizations, …
- the list is endless

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THE END